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# ASSET DEMANDS WITHOUT THE INDEPENDENCE AXIOM

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An important application of the theory of choice under uncertainty is to asset markets, and an important property in these markets is a preference for portfolio diversification. If an investor is an expected utility maximizer, then (s)he is risk averse if and only if (s)he exhibits a preference for diversification. This paper examines the relationship between risk aversion and portfolio diversification when preferences over probability distributions of wealth do not have an expected utility representation. Although risk aversion is not sufficient to guarantee a preference for portfolio diversification, it is necessary. Quasiconcavity of the preference functional (over probability distributions of wealth) together with risk aversion does imply a preference for portfolio diversification.

KEYWORDS: Portfolio diversification, risk aversion, independence axiom.

## 1. INTRODUCTION

AN IMPORTANT APPLICATION of the theory of choice under uncertainty is to markets of risky assets. If an investor is an expected utility maximizer, then (s)he is risk averse if and only if (s)he exhibits a preference for portfolio diversification. The property of having a preference for portfolio diversification is interesting in its own right, and also because a preference for diversification is equivalent to quasiconcave preferences over assets. The latter is useful for showing that the demand for assets is continuous, and more generally for second order optimality conditions to be satisfied. In this paper we examine the relationship between risk aversion and portfolio diversification when preferences over probability distributions of wealth do not have an expected utility representation. The results, corresponding to Propositions 1–3 below, are roughly as follows. First, risk aversion is not sufficient to guarantee a preference for portfolio diversification. However, a preference for diversification does imply risk aversion. Finally, quasiconcavity of the preference functional (over probability distributions of wealth) together with risk aversion does imply a preference for portfolio diversification (although quasiconcavity is not a necessary condition). These results may be contrasted with the fact that many other important characterizations of risk aversion do hold for general (non-expected-utility) preferences. In particular, Proposition 1 provides an example of a standard result from the theory of expected utility which cannot be extended to more general preferences by replacing the independence axiom with the assumption of differentiability (which is the approach used by Machina (1982a)).

We can relate these results to the classic work of Tobin (1957–58), which discusses diversification and risk aversion for the case of preferences over means and variances of distributions, $U(\mu, \sigma^2)$. A risk averter is defined as having a positive tradeoff between these two moments, that is an upward sloping indif-

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ference curve, and a "plunger" (i.e. nondiversifier) is a risk averter with quasiconvex preferences (over $(\mu, \sigma^2)$). Tobin noted that: "if the category defined as *plungers*... exists at all, their indifference curves must be determined by some process other than those described in 3.3" (Tobin (1957–8, p. 77)), where Section 3.3 derived mean-variance preferences from expected utility preferences with either normal distributions or quadratic Bernoulli utility functions. Our first result constructs an example which shows how preferences exhibiting risk aversion and plunging can be derived from general preferences over distributions of wealth. Our final result shows that a necessary condition for plungers is the failure of quasiconcavity of the preferences over distributions of wealth (although quasiconvexity isn't sufficient as in the mean-variance case).

## 2. THE PREFERENCES

Let $V\colon D \to \mathbb{R}$ be a preference function over the space of probability distributions on $[0,1]$, which is continuous in the topology of weak convergence and is consistent with first order stochastic dominance. The random variables $x^i$ $(i \geqslant 1)$ on the probability space $([0,1], B, \lambda)$ (where $B$ is the Borel field on the unit interval and $\lambda$ is the Lebesgue measure) have cumulative distribution functions $F(x^i; \cdot)$ which are also denoted $F^i$. Also, for any $n$ assets $x^i$, $i = 1, \ldots, n$, define the diversified asset $x^\alpha$ by $x^\alpha(s) \equiv \Sigma \alpha^i x^i(s)$ for every $s$, where $\alpha^i \geqslant 0$ and $\Sigma \alpha^i = 1$. $F^\alpha$ denotes the distribution $F(x^\alpha; \cdot)$ induced by the diversification, while $\alpha \cdot F$ is the convex combination (i.e. probability mixture) of the distributions, that is $\alpha \cdot F \equiv \Sigma \alpha^i F^i$.

DEFINITION 1: $V$ exhibits *risk aversion* if: (i) $V(F) \geqslant V(G)$ whenever $G$ is a mean preserving spread of $F$, or (ii) $V[p\tilde{F} + (1-p)F] \leqslant V[p\tilde{F} + (1-p)\delta_{E(F)}]$.

(The distribution with point mass at $c$ is denoted by $\delta_c$, and $E$ is the expectation operator.) These two properties are equivalent for preferences which are consistent with the first order stochastic dominance and continuous (Chew and Mao (1985); see also Machina (1982a)). If $V$ is Fréchet differentiable then they are also equivalent to concavity of the local utility function $u(\cdot, F)$ (Machina (1982a)).[2]

DEFINITION 2: $V$ exhibits *diversification* if for any $n \geqslant 1$ and any random variables $x^i$, $i = 1, \ldots, n$:

$$V(F^1) = \cdots = V(F^n) \quad \text{implies} \quad V(F^\alpha) \geqslant V(F^1)$$

for all $\alpha \in [0,1]^n$ satisfying $\Sigma \alpha^i = 1$.

[2] The local utility function satisfies $\int u(\cdot, F)d(\tilde{F} - F) = \Psi(\tilde{F} - F, F)$ where the latter is the Fréchet differential of $V$ at $F$ in the direction of $\tilde{F}$. Roughly speaking $u(\cdot, F)$ is the Bernoulli utility function of the linear approximation to $V$ at $F$. The approximation exists by the assumption of differentiability, and its linearity implies that the expected utility axioms are satisfied so a utility function exists.

This definition simply says that an individual will want to diversify among a collection of assets all of which are ranked equivalently. The relationship between Definition 2 and other definitions of diversification in the literature is discussed in the concluding remarks.

## 3. RISK AVERSION AND DIVERSIFICATION

It is now shown that although the equivalence of the definitions of risk aversion in terms of (i) and (ii) extends to nonlinear preferences, a similar extension of the equivalence to diversification fails. This is done by constructing a counterexample. A Fréchet differentiable preference function, with concave local utility functions, for which there exist assets $x^1, x^2$ and an $\alpha$ such that $V(F^\alpha) < V(F^1) = V(F^2)$ is provided.

PROPOSITION 1: *There exist V's which do exhibit risk aversion but do not exhibit diversification.*

PROOF: First choose any two assets $x^1$ and $x^2$ with different means where neither second order stochastically dominates (SSD) the other. Assume that $E(F^1) > E(F^2)$. Now choose an increasing and concave $v$ such that $\int v\, dF^1 < \int v\, dF^2$. (Such a $v$ exists since $F^1$ does not SSD $F^2$). Affinely normalize $v$ so that the first integral equals $E(F^2)$ and the second integral equals $E(F^1)$. Clearly $\int dF^1 > \int dF^\alpha > \int dF^2$. For $\alpha^1$ sufficiently close to 1: $\int v\, dF^1 < \int v\, dF^\alpha < \int v\, dF^2$, where the first inequality follows from concavity of $v$ and the second from continuity of $v$. Choose an $\bar{\alpha}^1$ sufficiently close to 1 for which the last inequality holds and then choose an increasing and differentiable $g$ such that: $g[E(F^1)] + g[E(F^2)] > g(\int v\, dF^{\bar{\alpha}}) + g(\int dF^{\bar{\alpha}})$, where $\bar{\alpha} = (\bar{\alpha}^1, 1 - \bar{\alpha}^1)$. Let $V(F) = g(\int v\, dF) + g(\int dF)$. By construction $V$ does not exhibit diversification ($F^{\bar{\alpha}}$ is less preferred than $F^1$ which is indifferent to $F^2$). On the other hand the local utility functions of $V$ are $u(\pi, F) = g'(\int v\, dF)v(\pi) + g'(\int dF)\pi$ and are concave by construction so $V$ exhibits risk aversion. *Q.E.D.*

It was shown above that the sufficiency of risk aversion for diversification in the case of expected utility preferences does not extend to more general preferences. However the reverse implication, that is the necessity of risk aversion, does extend to general preferences.

PROPOSITION 2: *If V exhibits diversification, then V exhibits risk aversion.*

PROOF: If $V$ does not exhibit risk aversion then there exist $\tilde{F}$, $F$, and $t$ such that $V[(1-t)\tilde{F} + tF] > V[(1-t)\tilde{F} + t\delta_{E(F)}]$. First assume that $F$ is a simple distribution (i.e. with finite support) which assigns rational probabilities $p_k$ to outcomes $\pi_k$. Rewrite $F$ as an equal probability distribution assigning probability $1/m$ to $\bar{\pi}_1, \ldots, \bar{\pi}_m$. Let $y$ be a random variable with the distribution $\tilde{F}$. Now for $k = 1, \ldots, m$ define the following assets:

$$x^k(s) = \bar{\pi}_{[i+k]} \quad \text{if} \quad s \in \left( t\frac{i-1}{m}, t\frac{i}{m} \right],$$

$$x^k(s) = y\left( s - \frac{t}{1-t} \right) \quad \text{if} \quad s > t$$

(where $[i+k] = i+k$ modulo $m$). For each $k$, $x^k$ clearly has the distribution $(1-t)\tilde{F} + tF$. On the other hand $x^\alpha$ for $\alpha = (1/m, \ldots, 1/m)$ has the distribution $(1-t)\tilde{F} + t\delta_{E(F)}$. To conclude note that $V(F^k) = V[(1-t)\tilde{F} + tF] > V[(1-t)\tilde{F} + t\delta_{E(F)}] = V(F^\alpha)$. If the $p_k$ aren't rational then a similar construction gives assets with distributions arbitrarily close to $F$, which is sufficient since $V$ is continuous by assumption. Similarly if $F$ is not simple then consider a sequence of simple distributions $F_n \uparrow F$, where for $n$ sufficiently large $V[(1-t)\tilde{F} + tF_n] > V[(1-t)\tilde{F} + t\delta_{E(F_n)}]$ by continuity of $V$. *Q.E.D.*

## 4. QUASICONCAVITY OF $V$ IN $F$, RISK AVERSION, AND DIVERSIFICATION

We have seen that risk aversion is not a sufficient condition for quasiconcavity of the induced preferences over assets. Since the latter is an important assumption for the analysis of asset markets, it is of interest to find conditions which imply this property (and hence diversification).

PROPOSITION 3: *If $V$ is quasiconcave in $F$ and $V$ exhibits risk aversion, then $V$ exhibits diversification.*

PROOF: It is first shown that if $V$ exhibits risk aversion then $V[F^\alpha] \geqslant V[\alpha \cdot F]$ (see also Roell (1985, Appendix A)). Let $u$ be an arbitrary concave Bernoulli utility function. Then $\int u\, dF^\alpha = \int u[\Sigma \alpha^i x^i(s)]\, d\lambda(s) \geqslant \Sigma \alpha^i \int u[x^i(s)]\, d\lambda(s) = \Sigma \alpha^i \int u\, dF^i = \int u\, d(\alpha \cdot F)$. But since $u$ is an arbitrary concave function and $E(F^\alpha) = E(\alpha \cdot F)$ it follows from Rothschild and Stiglitz (1970) that $\alpha \cdot F$ is a mean preserving spread of $F^\alpha$, so $V(F^\alpha) \geqslant V(\alpha \cdot F)$. Now note that quasiconcavity of $V$ implies that $V(\alpha \cdot F) \geqslant \min\{V(F^i)\}$ which together with preceding observation implies diversification. *Q.E.D.*

REMARKS: (1) Rothschild and Stiglitz (1971) show that a risk averse individual with expected utility preferences will want to diversify (equally) among assets which are i.i.d. This follows from showing that given $n$ i.i.d. random variables with distributions $F^i = \tilde{F}$ then $F^\alpha$ is a mean preserving spread of $F^{\bar{\alpha}}$ where

$\bar{\alpha} = (1/n, \dots, 1/n)$. Therefore also $V(F^{\bar{\alpha}}) \geqslant V(F^i)$, so their diversification result does extend to general preferences (without requiring quasiconcavity of $V$ in $F$). That is, any risk averse individual will diversify (equally) among i.i.d. assets. However, when the assets are not identically distributed, but the individual does rank them identically (that is, $V(F^i) = V(F^j)$ for all $i$ and $j$), then more than just risk aversion is needed for diversification (Proposition 2) and quasiconcavity of $V$ in $F$ (in addition to risk aversion) is sufficient.

(2) The proof of Proposition 2 applied risk aversion in the form of condition (ii) of Definition 1, while in the proof of Proposition 3, only condition (i) of Definition 1 is needed. This means that (even for preferences where (i) and (ii) aren't equivalent) diversification implies a preference for substituting the mean for the risky component in any compound lottery, and an aversion to mean preserving spreads together with quasiconcavity of $V$ in $F$ implies diversification.

While Proposition 3 shows that quasiconcavity (together with risk aversion) is sufficient for diversification, the following example shows that it is not necessary. The example does help clarify the role of quasiconcavity of $V$, since in it quasiconcavity can be relaxed only by putting a lower bound on the risk aversion.

Consider, for simplicity, random variables $x^i$ which map into $[0,1]$. Given a concave, twice continuously differentiable, increasing $v$ (with $v'' \leqslant \varepsilon < 0$), define $\bar{V}(F) = g(\int v\, dF) + g(\int dF)$, where $g$ satisfies $0 < g''(c) \leqslant \inf_{\pi \in [0,1]}[-v''(\pi)]$ and $g'(c) \geqslant [\sup_{\pi \in [0,1]} v'(\pi)]^2 + 1 > 0$ for all $c$ in the range of $\int dF$ and $\int v\, dF$. The following two Lemmas imply that this $\bar{V}$ exhibits diversification even though it is not quasiconcave in $F$. Note that the convexity of $\bar{V}$ in $F$ will depend on the convexity of $g$. On the other hand the risk aversion coefficient for the local utility functions of $\bar{V}$ is equal to $v'/(1+v'')$, which is bound from below by $(g''/g')(1 + v') \geqslant (g''/g')$.

Lemma 1: *$\bar{V}$ is convex.*

Proof:

$$\begin{aligned}
&\bar{V}\left[\alpha F^1 + (1-\alpha) F^2\right] \\
&\quad = g\left[\alpha \int dF^1 + (1-\alpha)\int dF^2\right] + g\left[\alpha \int v\, dF^1 + (1-\alpha)\int v\, dF^1\right] \\
&\quad \leqslant \alpha\left(g\left[\int dF^2\right] + g\left[\int v\, dF^2\right]\right) + (1-\alpha)\left(g\left[\int dF^1\right] + g\left[\int v\, dF^2\right]\right) \\
&\quad = \alpha \bar{V}(F^1) + (1-\alpha)\bar{V}(F^2).
\end{aligned}$$

Lemma 2: *$\bar{V}$ exhibits diversification.*

PROOF: Let $\bar{V}(F^1) = \bar{V}(F^2)$. This implies $\bar{V}(F^\alpha) \geqslant \bar{V}(F^1)$, for $\alpha = (\alpha^1, 1-\alpha^1)$ with $\alpha^1 \in [0,1]$. To see this, consider $H(\alpha^1) \equiv V(F^\alpha)$ as a function of $\alpha^1$. Since by assumption $H(0) = H(1) = \bar{V}(F^1)$ it is sufficient to show that $H'' \leqslant 0$.

$$H' = g'\left(\int dF^\alpha\right)\int \left(x^1(s) - x^2(s)\right) d\lambda(s)$$

$$+ g'\left(\int v\, dF^\alpha\right)\int \left[v'(x^\alpha(s))\left(x^1(s) - x^2(s)\right)\right] d\lambda(s).$$

$$H'' = g''\left(\int dF^\alpha\right)\left\{\int \left[x^1(s) - x^2(s)\right] d\lambda(s)\right\}^2$$

$$+ g''\left(\int v\, dF^\alpha\right)\left\{\int \left[v'(x^\alpha(s))\left(x^1(s) - x^2(s)\right)\right] d\lambda(s)\right\}^2$$

$$+ g'\left(\int v\, dF^\alpha\right)\int \left[v''(x^\alpha(s))\left(x^1(s) - x^2(s)\right)^2\right] d\lambda(s)$$

$$\leqslant A\left\{g''\left(\int dF^\alpha\right) + B^2 g''\left(\int v\, dF^\alpha\right) + C g'\left(\int v\, dF^\alpha\right)\right\}$$

where $A = \int [x^1(s) - x^2(s)]^2\, d\lambda(s)$, $B = \sup_{\pi \in [0,1]} v'(\pi)$, and $C = \sup_{\pi \in [0,1]} v''(\pi)$. Recall that $0 < g'' \leqslant -C$ so the last line is in fact less than or equal to: $AC[-1 - B^2 + g'(\int v'\, dF^\alpha)]$. However, $g' \geqslant 1 + B^2$ so the last expression is in fact nonpositive. *Q.E.D.*

The intuition for this example, and in fact for the entire paper, can be seen as follows. The proof of Proposition 3 shows that $V(F^\alpha) \geqslant V(\alpha \cdot F) \geqslant V(F^i)$, where the first inequality follows from risk aversion, and the second from quasiconcavity of $V$ in $F$. The necessity of risk aversion was shown by finding $F^i$'s such that the second inequality held with equality because the assets had the same distribution, while the first was reversed from lack of risk aversion (since the equal proportion diversification among the assets gave the expected value of the distribution). That risk aversion alone was not sufficient was demonstrated by finding a case where the reversal of the second inequality through lack of quasiconcavity of $V$ in $F$ was "stronger" than the first inequality (which remained correct because of risk aversion). Finally in order to show that quasiconcavity is not necessary an example where $V$ is not quasiconcave was constructed in such a way that the risk aversion inequality is always "stronger" than the reversal of the second (quasiconcavity) inequality.

## 5. CONCLUDING REMARKS

We conclude with a discussion of the relationship between the definition of diversification used here (Definition 2) and some other definitions in the literature. It is common (see Tobin (1957–8, p. 74) and other papers cited below) to require a diversifier to have a *strict* preference for diversification. The analog of this would require the inequality $V(F^\alpha) \geqslant V(F^1)$ in Definition 2 to be strict for

$\alpha \in (0,1)$. This would not change the results in this paper other than to replace weak with strict inequalities throughout. In Chew and Mao (1985), Chew, Karni, and Saffra (1985), and Machina (1982a) a diversifier is defined to allow for conditional diversification also. This would be achieved in Definition 2 by requiring: for all $\tilde{F} \in D$ and $t \in [0,1)$, if $V[t\tilde{F}+(1-t)F^i] = V[t\tilde{F}+(1-t)F^1]$ for $i = 2,\ldots,m$, then $V[t\tilde{F}+(1-t)F^\alpha] \geqslant V[t\tilde{F}+(1-t)F^1]$. Propositions 1–3 would also hold for this definition. These three papers also consider only diversification between a risky asset and a riskless asset. For the purposes of this paper it is more natural to require diversification (or conditional diversification) among risky assets also, as in Machina (1982b). In Chew, Karni, and Saffra (1985) concavity of the induced preferences over assets (rather than quasiconcavity) is used. A version of Proposition 3 clearly holds with a definition of diversification using concavity—in the statement of the proposition the requirement that $V$ is quasiconcave should be replaced by concavity of $V$.

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