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# COMMON KNOWLEDGE WITH PROBABILITY 1*

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Two people, 1 and 2, are said to have common knowledge of an event if both know it, 1 knows that 2 knows it, 2 knows that 1 knows it, 1 knows that 2 knows that 1 knows it, and so on. This paper provides a Bayesian definition of common knowledge, that is, a definition in terms of beliefs (probability measures). The main result is an equivalence between this definition and a definition in terms of the $\sigma$-fields representing 1 and 2's information. To obtain this result the conditional probabilities must be *proper* and the $\sigma$-fields *posterior completed.*

## 1. Introduction

The idea of common knowledge is central to game theory and the economics of uncertainty and information. For example, the non-cooperative analysis of a game (with complete information) starts with the assumption that the structure of the game is common knowledge among the players. Intuitively speaking, two people 1 and 2 are said to have common knowledge of an event if both know it, 1 knows that 2 knows it, 2 knows that 1 knows it, 1 knows that 2 knows that 1 knows it, and so on.

Common knowledge was first formally defined by Aumann (1976). Aumann assumes that 1 and 2's private information is represented by a pair of partitions of some state space $\Omega$. Individual $i$ is said to know an event $A$ at some state of the world $\omega$ if the member of $i$'s information partition which contains $\omega$ is itself contained in $A$. Using this definition of what it means to know an event, Aumann shows that an event $A$ is common knowledge at $\omega$ if and only if $A$ contains the member of the meet (finest common coarsening) of 1 and 2's partitions that contains $\omega$.

An important restriction on the information structure in Aumann (1976) is that the join (coarsest common refinement) of 1 and 2's partitions is assumed

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to consist of non-null events. But many decision problems naturally call for a general – possibly uncountable – state space, in which case null events (in the join) must be permitted. This paper provides a definition of common knowledge in this more general situation. (Of course, the definition coincides with Aumann's when his applies.) The starting point is a 'Bayesian' definition of knowledge: to say a person knows an event $A$ at some state $\omega$ means that (s)he assigns $A$ posterior probability one at $\omega$. Having defined what it means for someone to know an event $A$ at $\omega$, one can go on to define common knowledge of $A$ at $\omega$. The main result in this paper is an equivalence between a definition of common knowledge in terms of beliefs and a definition in terms of the $\sigma$-fields representing 1 and 2's information.[1]

Apart from the intrinsic interest in defining common knowledge on an infinite state space, there is one fundamental issue which can *only* be addressed if we are able to define common knowledge on an infinite $\Omega$. In both Aumann (1976) and this paper, the information partitions and priors (i.e., the information structure on $\Omega$) are assumed to be common knowledge in an informal sense. We say 'in an informal sense' because the information structure is not an event in $\Omega$ and a formal mathematical definition of common knowledge applies only to events in $\Omega$. Of course, any mathematical theorems one proves on common knowledge – such as the equivalence result in this paper – are true whether or not it is assumed that the information structure is common knowledge, since the theorems hold regardless of interpretation. However, to interpret the theorems as statements about common knowledge, it is necessary to make the assumption that the information structure is common knowledge. As argued in Aumann (1976, 1987), if this assumption is not satisfied then the state space can (and should) be expanded. In Brandenburger and Dekel (1985) the appropriate expanded state space is found such that if common knowledge is defined on this space, then the assumption that the information structure is common knowledge is in a certain sense without loss of generality. The point is that this expanded state space is uncountable even if the underlying state space is finite.[2]

Bayesian decision theory suggests a definition of knowledge in terms of beliefs – to say a person knows an event means that (s)he assigns it posterior probability one. Another approach would be to say that a person knows an event if (s)he is informed that it occurs – this is a definition in terms of an information partition/$\sigma$-field. To obtain an equivalence between these two definitions, the two approaches have to be reconciled. This is achieved by

[1]Nielsen (1984) provides a different definition of common knowledge on a general state space using Boolean $\sigma$-algebras.

[2]The expanded state space is the product of the underlying space of uncertainty, $S$ say, and the spaces of all possible 'types' of 1 and 2. Following Harsanyi (1967–68) and Mertens and Zamir (1985), a type of person 1 (resp. 2) is an infinite hierarchy of beliefs – over $S$, over 2's (resp. 1's) beliefs over $S$, and so on. The type spaces are uncountable even if $S$ is finite.

assuming that the probability measures are regular and proper (see Definition 2.2) and that the $\sigma$-fields are completed in a suitable manner (see Definition 2.3).

Recall that Bayes' rule says nothing about how an individual $i$ updates his/her beliefs over $\Omega$ if informed that a null event occurs. Regularity says that $i$ must have a belief over $\Omega$ even if informed of some null partition cell $H$. But it is quite possible for $i$ to ignore the fact that (s)he was informed that the true state lies in $H$, i.e., to assign positive posterior probability to states outside of $H$. Properness requires that after being informed of any partition cell $H$, $i$ assigns posterior probability one to $H$ even if $H$ has prior probability zero. The use of the term 'properness' originated in Blackwell and Ryll-Nardzewski (1963). Blackwell and Ryll-Nardzewski (1963) and Blackwell and Dubins (1975) argue that an intuitively satisfactory theory of probability should involve proper regular conditional probabilities.

To get an idea of the role of completion of the $\sigma$-fields, suppose that $i$ has no information, i.e., has the trivial partition $\{\Omega\}$, but $i$ assigns probability one to a strict subset $A$ of $\Omega$. Then the only event of which $i$ is informed is $\Omega$, but $i$ knows $A$ according to the definition in terms of beliefs. What is needed here is to add into $i$'s partition the events to which $i$ assigns probability one or – what amounts to the same thing – the events to which $i$ assigns probability zero. That is, it is necessary to complete $i$'s partition.

## 2. Common knowledge in terms of beliefs

This section begins with a review of the definition of common knowledge in Aumann (1976). There is a measurable space $(\Omega, \mathscr{F})$, where $\Omega$ is the space of states of the world and $\mathscr{F}$ is a $\sigma$-field of subsets of $\Omega$. There are two individuals indexed by $i=1,2$. Individual $i$'s information about the state of the world is represented by a (measurable) partition $\mathscr{P}^i$ of $\Omega$. If the true state is $\omega$, then $i$ is informed of the member $\mathscr{P}^i(\omega)$ of $\mathscr{P}^i$ that contains $\omega$. 1 and 2 have a common prior on $\Omega$ which assigns positive probability to every event in the join (coarsest common refinement) $\mathscr{P}^1 \vee \mathscr{P}^2$ of 1 and 2's partitions.

Consider an event $A \in \mathscr{F}$ and a state of the world $\omega \in \Omega$. $i$ is said to know $A$ at $\omega$ if $\mathscr{P}^i(\omega) \subset A$. An event $A$ is said to be common knowledge at some state $\omega$ if 1 knows $A$ at $\omega$, 2 knows $A$ at $\omega$, 1 knows 2 knows $A$ at $\omega$, 2 knows 1 knows $A$ at $\omega$, and so on. Aumann shows that $A$ is common knowledge at $\omega$ if and only if $(\mathscr{P}^1 \wedge \mathscr{P}^2)(\omega) \subset A$ where $(\mathscr{P}^1 \wedge \mathscr{P}^2)(\omega)$ is the member of the meet (finest common coarsening) of $\mathscr{P}^1$ and $\mathscr{P}^2$ that contains $\omega$.

As argued in the introduction, many decision problems naturally call for an uncountable state space $\Omega$, in which case null events (in the join) must be allowed. Start with a probability space $(\Omega, \mathscr{F}, P)$ where $P$ is the common prior of the two individuals.[3] Individual $i$'s information is described by a sub

[3]All our results generalize immediately to the case of more than two individuals and to non-identical priors.

$\sigma$-field $\mathscr{F}^i$ of $\mathscr{F}$. For each $i$, fix a version of a regular conditional $P$-probability given $\mathscr{F}^i$, that is, a function $Q^i:\mathscr{F}\times\Omega\rightarrow[0,1]$ such that:

(1) for each $A\in\mathscr{F}, Q^i(A,\cdot)$ is a version of $P(A|\mathscr{F}^i)$;
(2) for each $\omega\in\Omega, Q^i(\cdot,\omega)$ is a probability measure on $\mathscr{F}$.

(This can be done if, for example, $\Omega$ is a complete separable metric space and $\mathscr{F}$ is the Borel field on $\Omega$.) This completes the description of the information structure. The elements of this structure – the probability space $(\Omega,\mathscr{F},P)$, the $\sigma$-fields $\mathscr{F}^1$, $\mathscr{F}^2$, and the conditional probabilities $Q^1$, $Q^2$ – are assumed to be common knowledge between 1 and 2. In particular, notice that the conditional probabilities $Q^1$, $Q^2$ must be specified and hence must be common knowledge – it is not enough for just the prior $P$ to be common knowledge. As was discussed in the introduction, this assumption that the information structure is common knowledge is in a certain sense without loss of generality.

Consider an event $A\in\mathscr{F}$ and a state of the world $\omega\in\Omega$. Define '$i$ knows $A$ at $\omega$' to mean that $i$ assigns posterior probability 1 to $A$ at $\omega$, i.e., $Q^i(A,\omega)=1$. The event that $i$ knows $A$, to be denoted $K^i(A)$, is then the set of $\omega$'s such that $i$ knows $A$ at $\omega$:

$$K^i(A)=\{\omega:Q^i(A,\omega)=1\}.$$

$K^i(\cdot)$ is a function from $\mathscr{F}$ to $\mathscr{F}^i$. The following properties of $K^i(\cdot)$ show that this function captures some aspects of one's intuitive notion of what 'to know' means. (The proofs are straightforward and are omitted.)

*P.1.* For any $A\in\mathscr{F}$, $K^i(A)\in\mathscr{F}^i$.

*P.2.* For any $A,B\in\mathscr{F}$, if $A\subset B,[i]$, then $K^i(A)\subset K^i(B)$.

*P.3.* For any $A_1,A_2,\ldots\in\mathscr{F}$, $K^i(\bigcap_{n=1}^{\infty}A_n)=\bigcap_{n=1}^{\infty}K^i(A_n)$.

The notation $A\subset B$, $[i]$ means that for every $\omega\in\Omega$, $Q^i(A-B,\omega)=0$. So if $A\subset B,[i]$, then $i$'s posterior belief at every state of the world $\omega$ is that $B$ happens whenever $A$ happens. P.2 says that in this case if $i$ knows $A$ then $i$ knows $B$. Note that $A\subset B$ implies $A\subset B,[i]$, which in turn implies that $P(A-B)=0$ (which says that $i$'s *prior* belief is that $B$ happens whenever $A$ happens), but the converses do not hold. P.2 has the form of the subsequent results in this paper in that it uses the conditionals $Q^i$ and not the prior $P$. P.3 says that $i$ knows $A_1$ and $A_2$ and so on if and only if $i$ knows $A_1$ and $i$ knows $A_2$ and so on.

Now consider the event: 1 knows $A$, 1 knows 2 knows $A$, 1 knows 2 knows 1 knows $A$, and so on. Call this event $L^1A$. Formally,

$$L^1A = K^1A \cap K^1K^2A \cap K^1K^2K^1A \cap \cdots.$$

(Note that by P.1 all sets of the type $K^1K^2 \ldots A$ lie in $\mathscr{F}^1$. Therefore $L^1A$, being a countable intersection of such sets, also lies in $\mathscr{F}^1$.) Let $L^2A$ denote the corresponding event: 2 knows $A$, 2 knows 1 knows $A$, and so on.

*Definition 2.1.* An event $A \in \mathscr{F}$ is common knowledge at a state of the world $\omega \in \Omega$ if $\omega \in L^1A \cap L^2A$.

Definition 2.1 formalizes the notion of common knowledge using 1 and 2's beliefs, i.e., their posteriors $Q^1$, $Q^2$, as Bayesian decision theory suggests. An 'informational' aproach would suggest that common knowledge can be defined using the $\sigma$-fields $\mathscr{F}^1$, $\mathscr{F}^2$, i.e., using 1 and 2's private information. The objective now is to relate Definition 2.1 to an informational definition. To see the first issue which arises, consider the following simple example (see fig. 1).[4]

*Example.* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathscr{P}^1 = \mathscr{P}^2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$, $A = \{\omega_3\}$. $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$, $P(\{\omega_3\}) = 0$. If 1 and 2 are informed of $\{\omega_3\}$, then they ignore this, so that $P(\cdot \mid \{\omega_3\}) = P(\cdot)$. Consider the state $\omega_3$. It looks like $A$ should be common knowledge at $\omega_3$ since the member of the meet of $\mathscr{P}^1$ and $\mathscr{P}^2$ that contains $\omega_3$, namely $\{\omega_3\}$, is contained in $A$. But $A$ is not common knowledge at $\omega_3$ in the sense of Definition 2.1 since $K^1A = K^2A = \emptyset$.

In this example, if 1 and 2 are informed of $\{\omega_3\}$, they ignore this information. To rule out this somewhat implausible situation one should require that an individual assign posterior probability one to *any* partition cell, even if that cell has zero prior probability. Another justification for

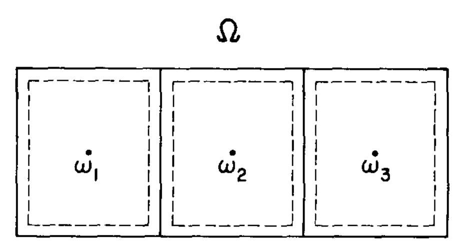


Fig. 1

[4]The examples in the paper use a finite $\Omega$. Although with finite $\Omega$ and common priors one can simply throw out the null events, this procedure does not extend to the infinite case. The examples are designed to be illustrative of the difficulties in the general case.

imposing this restriction is that if it does not hold, an individual may not know his/her own beliefs: when informed of $\{\omega_3\}$, 1 and 2 assign posterior probability $\frac{1}{2}$ to $\{\omega_1\}$ (and hence to the belief $P(\cdot|\{\omega_1\})$) and posterior probability $\frac{1}{2}$ to $\{\omega_2\}$ (and hence to the belief $P(\cdot|\{\omega_2\})$). The general version of the restriction on conditional probabilities we want to impose is called properness [see Blackwell and Ryll-Nardzewski (1963) and Blackwell and Dubins (1975)].

*Definition 2.2.* $Q^i$ is proper if for each $\omega \in \Omega$, $Q^i(F,\omega) = \mathbf{1}_F(\omega)$ for every $F \in \mathscr{F}^i$.

$\mathbf{1}_F(\cdot)$ denotes the indicator function. Theorem 1 in Blackwell and Ryll-Nardzewski (1963) provides a necessary and sufficient condition for the existence of a proper version. For our purposes there is no loss of generality in assuming that $Q^i$ is proper. [This is because on the expanded state space in Brandenburger and Dekel (1985), properness is automatically satisfied provided the underlying state space is complete separable metric.] The assumption of properness is also made in the literature on extensive form refinements of Nash equilibrium: it is implicit in the intuition behind subgame perfection [Selten (1965)] and in the definition of a sequential equilibrium [Kreps and Wilson (1982)]. Axiomatic characterizations of proper conditional probabilities can be found in Brandenburger and Dekel (1986) and Myerson (1986). Given properness, the function $K^i(\cdot)$ defined earlier can be shown to satisfy the following properties in addition to P.1–P.3.

*P.4.* For any $A \in \mathscr{F}$, $K^iA \subset A$, $[i]$.

*P.5.* For any $F \in \mathscr{F}^i$, $K^iF = F$.

Under the assumption that $Q^1$, $Q^2$ are proper we can now prove a one-way implication between the definition of common knowledge in terms of beliefs (Definition 2.1) and an informational definition in terms of the $\sigma$-fields $\mathscr{F}^1$ and $\mathscr{F}^2$.

*Lemma 2.1. Suppose there is a set $F$ in the meet $\mathscr{F}^1 \wedge \mathscr{F}^2$ such that $\omega \in F$ and $F \subset A$, $[i]$, $i=1,2$. Then $A$ is common knowledge at $\omega$.*

*Proof.* If $F \subset A$, [1], then $K^1F \subset K^1A$ by P.2. But by P.5, $F = K^1F$, so $F \subset K^1A$. Hence $\omega \in K^1A$. Similarly, $K^2F \subset K^2A$ by P.2 and $F = K^2F$ by P.5. So $F \subset K^2A$ and thus $F = K^1F \subset K^1K^2A$ by P.5 and P.2. Hence $\omega \in K^1K^2A$. Continuing in this way shows that $\omega \in L^1A$. A similar argument shows that $\omega \in L^2A$. □

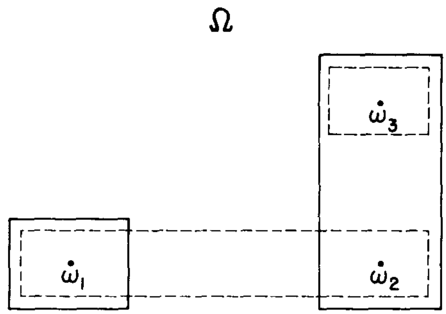


Fig. 2

In order to obtain a converse to Lemma 2.1, and hence an equivalence between Definition 2.1 and an informational definition of common knowledge, the $\sigma$-fields $\mathscr{F}^1$, $\mathscr{F}^2$ must be completed as the following example illustrates (see fig. 2).

*Example.* $\Omega=\{\omega_1,\omega_2,\omega_3\}$, $\mathscr{P}^1=\{\{\omega_1\},\{\omega_2,\omega_3\}\}$, $\mathscr{P}^2=\{\{\omega_1,\omega_2\},\{\omega_3\}\}$, $A=\{\omega_1\}$. $P(\{\omega_1\})=P(\{\omega_3\})=\frac{1}{2}$, $P(\{\omega_2\})=0$, $K^1A=\{\omega_1\}$, $K^2A=\{\omega_1,\omega_2\}$, $K^1K^2A=\{\omega_1\}=K^1A$. Continuing in this way shows that $L^1A=\{\omega_1\}$ and $L^2A=\{\omega_1,\omega_2\}$, so $\omega_1\in L^1A\cap L^2A$. That is, $A$ is common knowledge at $\omega_1$ in the sense of Definition 2.1. On the other hand, the meet of 1 and 2's partitions is just the trivial partition $\{\Omega\}$. Hence the converse to Lemma 2.1 fails since it is false that $\Omega\subset A$, $[i]=1,2$.

It looks like the way to deal with the problem in this example is to throw out the null event $\{\omega_2\}$. But this cannot be the right intuition for a general (infinite) $\Omega$. Instead let's do the opposite – *add in* the null events to $i$'s partition. This procedure is known in probability theory as completion [Chung (1974, p. 31, Exercise 20)]. However, the standard notion of completion, which would add in all the events to which $i$ assigns *prior* probability 0, is inappropriate.[5] Only those events to which $i$ assigns posterior probability 0 at *every* state of the world should be added to $i$'s partition. We call this procedure 'posterior completion'.

*Definition 2.3.* The posterior completion of $\mathscr{F}^i$ is the $\sigma$-field $\hat{\mathscr{F}}^i$ generated by $\mathscr{F}^i$ and the class of sets $\{G\in\mathscr{F}:Q^i(G,\omega)=0$ for every $\omega\in\Omega\}$.

Lemma 2.2 below provides a useful characterization of $i$'s posterior completed $\sigma$-field $\hat{\mathscr{F}}^i$. It says that $\hat{\mathscr{F}}^i$ contains all the events $G$ in the underlying

[5]To see why, refer back to the second example with $P(\{\omega_1\})=1$ and $P(\{\omega_2\}|\{\omega_2,\omega_3\})=P(\{\omega_3\}|\{\omega_2,\omega_3\})=\frac{1}{2}$. It would be wrong to add $\{\omega_2\}$, to which $P$ assigns prior probability 0, into 1's partition since there is no $\omega$ at which 1 knows $\{\omega_2\}$.

$\sigma$-field $\mathscr{F}$ such that, whatever state of the world occurs, $i$ knows either $G$ or the complement of $G$.

*Lemma 2.2.* $\hat{\mathscr{F}}^i = \{G \in \mathscr{F}$: *for every* $\omega \in \Omega, Q^i(G, \omega) = 0$ *or* $1\}$.

The proof of Lemma 2.2 relies on standard arguments on completion and is omitted. Using Lemma 2.2 a partial converse to Lemma 2.1 can now be stated.

*Lemma 2.3. Suppose $A$ is common knowledge at $\omega$. Then there is a set $F$ in the meet $\hat{\mathscr{F}}^1 \wedge \hat{\mathscr{F}}^2$ of the posterior completed $\sigma$-fields such that $\omega \in F$ and $F \subset A$, $[i] = 1, 2$.*

*Proof.* Set $F = L^1A \cap L^2A$. Clearly $L^1A \cap L^2A \subset L^1A \subset K^1A$, and P.4 says that $K^1A \subset A$, [1]. So $L^1A \cap L^2A \subset A$, [1]. We now want to show that $L^1A \cap L^2A \in \hat{\mathscr{F}}^1$, i.e., that for every $\omega$, $Q^1(L^1A \cap L^2A, \omega) = 0$ or 1 (using the characterization in Lemma 2.2). By definition if $\omega \in K^1(L^1A \cap L^2A)$, $Q^1(L^1A \cap L^2A, \omega) = 1$. So to prove that $L^1A \cap L^2A \in \hat{\mathscr{F}}^1$ it will be enough to show that if $\omega \in \Omega - K^1(L^1A \cap L^2A)$, $Q^1(L^1A \cap L^2A, \omega) = 0$. But $K^1(L^1A \cap L^2A) = K^1L^1A \cap K^1L^2A = L^1A$ by P.3 and P.5. Hence $L^1A \cap L^2A \subset K^1(L^1A \cap L^2A)$. If $\omega \in \Omega - K^1(L^1A \cap L^2A)$, then $Q^1[\Omega - K^1(L^1A \cap L^2A), \omega] = 1$ by properness. So if $\omega \in \Omega - K^1(L^1A \cap L^2A)$, $Q^1(L^1A \cap L^2A, \omega) = 0$ as required. Similar arguments establish that $L^1A \cap L^2A \subset A, [2]$, and $L^1A \cap L^2A \in \hat{\mathscr{F}}^2$. □

There is one more difficulty to overcome before the equivalence result can be stated. The remaining problem is that when the $\sigma$-fields are completed we get events in the meet which are believed never to happen. To see this, refer back to the second example with $P(\{\omega_1\}) = P(\{\omega_3\}) = \frac{1}{2}$. $\{\omega_2\}$ is a member of both 1 and 2's posterior completed $\sigma$-fields, hence it lies in the meet. But even if $\{\omega_2\}$ happens, 1 and 2 both assign posterior probability zero to $\{\omega_2\}$. So certainly $\{\omega_2\}$ is not common knowledge at any state of the world in the sense of Definition 2.1. In order to rule out such situations it is necessary to consider only 'non-null' members of the meet: say an event $G \in \mathscr{F}$ is non-null if for $i = 1, 2$, $Q^i(G, \omega) > 0$ for every $\omega \in G$.

*Proposition 2.1. $A$ is common knowledge at $\omega$ if and only if there is a non-null $F \in \hat{\mathscr{F}}^1 \cap \hat{\mathscr{F}}^2$ such that $\omega \in F$ and $F \subset A$, $[i]$, $i = 1, 2$.*

*Proof. Only if.* Set $F = L^1A \cap L^2A$ and proceed as in the proof of Lemma 2.3. The only additional step is to show that $F$ is non-null. But this follows immediately from $L^1A \cap L^2A \subset K^i(L^1A \cap L^2A)$, $i = 1, 2$, which was shown in the course of that proof.

*If.* Suppose $\omega \in F$ and $F \subset A$, $[i]$, $i = 1, 2$, where $F$ is a non-null member of the meet. Since $F$ is non-null, Lemma 2.2 implies that $Q^i(F, \omega) = 1$ for every $\omega \in F$. That is, $F \subset K^i F$, $i = 1, 2$. The proof now follows exactly the lines of the proof of Lemma 2.1. □

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