GAMES AND ECONOMIC BEHAVIOR 4, 182–201 (1992)

Correlated Equilibrium with Generalized Information Structures*

ADAM BRANDENBURGER

Harvard Business School, Boston, Massachusetts 02163

EDDIE DEKEL

Department of Economics, University of California, Berkeley, California 94720

AND

JOHN GEANAKOPLOS

Cowles Foundation, Yale University, New Haven, Connecticut 06520

Received August 29, 1989

We study an extension of the notion of correlated equilibrium that allows players to make information processing errors. A typical way in which such errors arise is if players take information at face value, that is, if players do not consider how their information would differ in different states of the world. (This is despite the fact that such considerations could well yield better information.) We model errors such as these by weakening the assumption that players possess information partitions. It is shown that introducing information processing errors is equivalent to allowing "subjectivity," i.e., differences between the players' priors. Hence a bounded rationality justification of subjective priors is provided. We examine in detail the implications of allowing various forms of information processing errors. © 1992 Academic Press, Inc.

* We are indebted to Don Brown for helpful discussions on related work and to Kim Border and a referee for useful comments. Financial support from the Harvard Business School Division of Research, the Miller Institute for Basic Research in Science, NSF Grant SES-8808133 and the Sloan Foundation is gratefully acknowledged.

0899-8256/92 $3.00

1. Introduction

It is customary in game theory to model a situation of differently informed players in terms of partitions of a state space. In this paper we study the correlated equilibria (Aumann, 1974, 1987) of games in which players make information processing errors. To do this we replace partitions by more general information structures called possibility correspondences. As a result our players can ignore bad news, be unaware of events they do not observe, forget, or even fail to imagine some contingencies. Possibility correspondences have been examined by Shin (1986, 1987), Samet (1990), and Geanakoplos (1989). This last paper also introduced the notion of Nash equilibrium for games in which players make information processing errors.

In a correlated equilibrium, information matters to a player only insofar as it provides a clue to other players' choices. (The states of the world, which describe the players' uncertainty, do not directly enter any player's payoff function.) Hence the information processing errors allowed for by possibility correspondences cause players to make mistakes (in an indirect fashion) about other players' actions. We examine the set of correlated equilibria obtained by varying the set of states of the world and the players' priors and possibility correspondences. This enables us to compare the set of (generalized) correlated equilibria with information processing errors, with the set of (conventional) correlated equilibria in which such errors are absent. In the Nash equilibria studied in Geanakoplos (1989), the states of the world *do* enter the players' payoff functions. Hence the set of states of the world and the players' priors and information structures are naturally thought of as fixed. Moreover, in this context, information processing errors play an additional role: they directly affect a player's payoffs through mistakes about the state of the world.

In comparing the sets of correlated equilibria—with and without information processing errors—we adopt two approaches. In the first, the perspective is that of the players themselves, that is, we focus on the players' strategies and payoffs. We show that any correlated equilibrium with information processing errors is, from the viewpoint of the players, decision-theoretically equivalent to some (subjective) correlated equilibrium in which such errors are absent but players may have different prior beliefs (Proposition 4.1). Conversely, we prove that any subjective correlated equilibrium is decision-theoretically equivalent to a correlated equilibrium with common priors, but with (significant) information processing errors (Proposition 4.2). Together these results establish that information processing errors and different priors are interchangeable as far as the players' decision problems are concerned. Moreover we show that allowing players to make further errors in the calculation of conditional probabilities adds nothing new to the analysis: such miscalculations can already be subsumed in information processing errors (Remark 4.1).

In the second approach, the perspective is that of an outside observer. We suppose the players share a common (objective) prior and we focus on the distribution on actions induced by a correlated equilibrium. When the players have partitions (i.e., make no mistakes) and share a common prior, the set of correlated equilibrium distributions on actions is a closed, convex set (see Aumann, 1974, 1987). Permitting mistakes, but keeping a common prior, must maintain or enlarge the set of correlated equilibrium distributions. We describe a class of information processing errors which nevertheless leaves the set of correlated equilibrium distributions unchanged (Proposition 5.2). We also characterize the set of correlated equilibrium distributions that arise when we allow a larger class of mistakes by the players (Proposition 5.1). This latter set is again convex, but not necessarily closed.

The organization of the rest of the paper is as follows. Section 2 describes alternative information structures. Section 3 defines generalized correlated equilibria and discusses their interpretation. Section 4 establishes the results on decision-theoretic equivalence. Section 5 characterizes generalized correlated equilibrium distributions.

2. Alternative Information Structures

The information structures discussed in this section all start with a finite set Ω of possible states of the world. In the standard framework, player i's information is represented by a partition $\mathbb{H}^i$ of Ω, that is, a class of nonempty disjoint subsets of Ω that covers Ω. Given a partition $\mathbb{H}^i$, define a correspondence H^i: $\Omega \to 2^\Omega \setminus \{\emptyset\}$ by letting $H^i(\omega)$ be the member of $\mathbb{H}^i$ that contains ω. (Clearly the range of H^i is then $\mathbb{H}^i$.) If the true state is ω, player i is informed of $H^i(\omega)$. A more general way of representing information that allows for information processing errors is via a *possibility correspondence* P^i: $\Omega \to 2^\Omega \setminus \{\emptyset\}$. The interpretation is that if the true state is ω, player i regards all states in $P^i(\omega)$ as possible. Possibility correspondences have been studied by Shin (1986, 1987), Samet (1990), and Geanakoplos (1989). In this paper we shall make use of various combinations of the following properties of the correspondence P^i.

(1) *(Nondelusion).* For all $\omega \in \Omega$, $\omega \in P^i(\omega)$.

(2) *(Knowing That You Know, KTYK).* For all $\omega \in \Omega$, $\omega' \in P^i(\omega)$ implies $P^i(\omega') \subset P^i(\omega)$.

To define the properties of balancedness and positive balancedness, we need a preliminary definition. Say a set $E \subset \Omega$ is *self-evident* if for every

$\omega \in E$, $P^i(\omega) \subset E$. That is, E is self-evident if whenever E happens, i knows that E happens (i can only imagine states in which E happens). Given a possibility correspondence P^i: $\Omega \rightarrow 2^\Omega \setminus \{\emptyset\}$, let $\mathbb{P}^i$ denote the range of P^i.

(3) *(Balancedness).* For every self-evident set $E \subset \Omega$ there is a function β: $\mathbb{P}^i \rightarrow \mathbb{R}$ such that

$$\chi_E = \sum_{\substack{R^i \in \mathbb{P}^i \\ R^i \subset E}} \beta(R^i)\chi_{R^i},$$

where χ_A denotes the characteristic function of A (i.e., $\chi_A(\omega) = 0$ or 1 according as $\omega \notin A$ or $\omega \in A$).

(4) *(Positive Balancedness).* For every self-evident set $E \subset \Omega$ there is a function β: $\mathbb{P}^i \rightarrow \mathbb{R}_+$ such that

$$\chi_E = \sum_{\substack{R^i \in \mathbb{P}^i \\ R^i \subset E}} \beta(R^i)\chi_{R^i}.$$

Property (1) of P^i says that player i always imagines the true state to be possible. Property (2) says that if i knows some set A at ω, and can imagine ω', then he would know A at ω'. In other words, i knows what he knows. It can be shown (see Geanakoplos, 1989) that, assuming nondelusion, KTYK implies balancedness. In fact, in the context of certain correlated equilibria with information processing errors, balancedness is no more general than KTYK (see Proposition 5.1 and Remark 5.2). Clearly, positive balancedness is more restrictive than balancedness but weaker than assuming a partition. (Also, positive balancedness neither implies nor is implied by KTYK.) Nevertheless, we show that for certain correlated equilibria with information processing errors, positive balancedness is equivalent to assuming a partition (see Proposition 5.2). For a discussion of the kinds of information processing errors captured by possibility correspondences satisfying various combinations of Properties (1)–(4), see Geanakoplos (1989).

It remains to discuss the issue of player i's beliefs. The usual Bayesian approach is to assume that i has, in addition to a partition $\mathbb{H}^i$ of Ω, a prior probability distribution π^i on Ω. If the true state is ω, the probability that i assigns to a set $A \subset \Omega$ is then given by the conditional probability $\pi^i(A|H^i(\omega))$. Likewise, when i has a possibility correspondence P^i, it is natural to suppose that if the true state is ω then i assigns probability $\pi^i(A|P^i(\omega))$ to A. We could also imagine allowing for mistakes in computing probabilities by supposing that player i, rather than calculating conditionals as just described, possesses a more general "belief function"

δ^i: $\Omega \to \Delta(\Omega)$ giving i's beliefs at each state of the world. (Here $\Delta(\Omega)$ denotes the set of all probability measures on Ω.) The probability that i assigns to an event $A \subset \Omega$ when ω occurs, $\delta^i(\omega)(A)$, might not be obtained by taking conditionals with respect to a prior π^i and possibility correspondence P^i, that is, $\delta^i(\omega)(A) \neq \pi^i(A|P^i(\omega))$. For example, player i might miscalculate conditional probabilities. In fact, we can show that this extra generality adds nothing new: errors in calculating probabilities can be captured in information processing errors (see Remark 4.1).

3. Generalized Correlated Equilibrium

This section begins with a review of the usual notion of correlated equilibrium as introduced by Aumann (1974). Consider an n-person game $\Gamma = \langle A^1, \ldots, A^n; u^1, \ldots, u^n \rangle$ where, for each $i = 1, \ldots, n$, A^i is player i's finite set of actions and u^i: $\times_{j=1}^n A^j \to \mathbb{R}$ is i's payoff function. For any finite set Y, let $\Delta(Y)$ denote the set of probability measures on Y. Given sets $Y^1, \ldots, Y^n$, Y^{-i} will denote the set $Y^1 \times \cdots \times Y^{i-1} \times Y^{i+1} \times \cdots \times Y^n$, and $y^{-i} = (y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^n)$ a typical element of Y^{-i}. To define a correlated equilibrium of Γ, one must add to the basic description of the game a finite state space Ω and, for each i, a prior π^i on Ω, a partition $\mathbb{H}^i$ of Ω, and a map f^i: $\Omega \to A^i$ satisfying $H^i(\omega') = H^i(\omega)$ implies $f^i(\omega') = f^i(\omega)$. A *correlated equilibrium* (CE) of Γ is a collection $\langle \Omega; \pi^i, H^i, f^i \rangle$ where for every i and each $\omega \in \Omega$ the conditional expected payoff to i of $f^i(\omega)$ is at least as great as the conditional expected payoff to i of any other action a^i:

$$\sum_{\omega' \in H^i(\omega)} \pi^i(\omega'|H^i(\omega))u^i(f^i(\omega), f^{-i}(\omega')) \geq \sum_{\omega' \in H^i(\omega)} \pi^i(\omega'|H^i(\omega))u^i(a^i, f^{-i}(\omega'))$$

for all $a^i \in A^i$.[1] If all the π^i's are the same (the Common Prior Assumption) then the CE is an *objective correlated equilibrium* (OCE). If we wish to emphasize the possibility of different priors, we will refer to a CE as a *subjective correlated equilibrium* (SCE).[2]

A *generalized correlated equilibrium* (GCE) is exactly the same as a CE, except that the players have possibility correspondences P^i in place of partitions $\mathbb{H}^i$. Thus $\langle \Omega; \pi^i, P^i, f^i \rangle$ is a GCE of Γ if for every i and each $\omega \in \Omega$:

[1] The conditional distributions $\pi^i(\cdot|H^i(\omega))$ are assumed to exist for every $H^i(\omega)$, even if $\pi^i(H^i(\omega)) = 0$, and to satisfy $\pi^i(H^i(\omega)|H^i(\omega)) = 1$ (i.e., properness in the sense of Blackwell and Dubins (1975)).

[2] Strictly speaking, our definition is that of an a posteriori equilibrium (Aumann, 1974, Sect. 8) since optimality on *every* $H^i(\omega)$ is required.

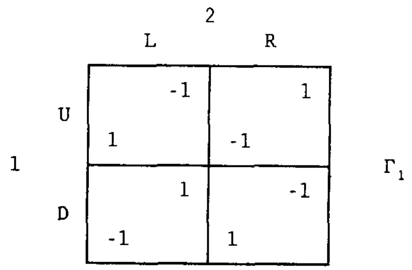

FIGURE 1

$$P^i(\omega') = P^i(\omega) \text{ implies } f^i(\omega') = f^i(\omega); \tag{1}$$

$$\sum_{\omega' \in P^i(\omega)} \pi^i(\omega' \mid P^i(\omega)) u^i(f^i(\omega), f^{-i}(\omega')) \geq \sum_{\omega' \in P^i(\omega)} \pi^i(\omega' \mid P^i(\omega)) u^i(a^i, f^{-i}(\omega')) \tag{2}$$

for all $a^i \in A^i$. If all the π^i's are the same, we refer to an *objective generalized correlated equilibrium* (OGCE). If we wish to emphasize the possibility of different priors, we will refer to a GCE as a *subjective generalized correlated equilibrium* (SGCE).

We illustrate the definition of a GCE by means of two examples. Consider first the familiar game of Matching Pennies depicted in Fig. 1.

Recall that in any OCE of Γ_1 the conditional expected payoffs to the players are always 0 (see Aumann, 1974). By contrast, we now describe an OGCE of Γ_1 in which the conditional expected payoffs to each player are all strictly positive.

Figure 2 depicts the state space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and the maps f^1 and f^2, and illustrates the possibility correspondences P^1 and P^2. Player 1's possibility correspondence P^1 satisfies: $P^1(1) = \{1\}$, $P^1(2) = \{1, 2\}$, $P^1(3) = \{1, 3\}$, $P^1(4) = \{4, 6\}$, $P^1(5) = \{5, 6\}$, $P^1(6) = \{6\}$. Player 2's possibility correspondence P^2 satisfies: $P^2(1) = P^2(4) = P^2(5) = \{1, 4, 5\}$, $P^2(2) = P^2(3) = P^2(6) = \{2, 3, 6\}$. Finally, the common prior π assigns probability $\frac{1}{5}$ to each of states 1 and 6, probability $\frac{3}{20}$ to each of states 2, 3, 4, and 5. It is readily verified that $\langle \Omega; \pi, P^i, f^i \rangle$ is an OGCE of Γ_1.

The conditional expected payoffs to player 1 are either $\frac{1}{7}$ or 1; the conditional expected payoff to player 2 is always $\frac{1}{5}$. Note that the possibility correspondence P^1 satisfies nondelusion and KTYK (hence balancedness), but is not positively balanced. In Section 4 we define a notion of decision-theoretic equivalence between GCEs. This definition permits a general characterization of conditional expected payoffs that arise in GCEs in terms of those arising in CEs.

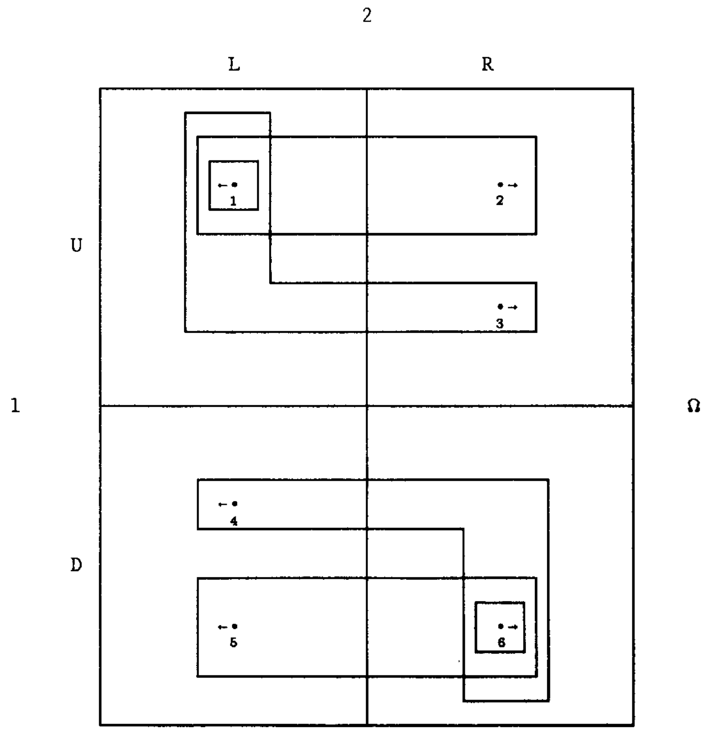

FIGURE 2

In this example, the distribution on actions assigns probability $\frac{1}{5}$ to each of (U, L) and (D, R), and probability $\frac{3}{10}$ to each of (D, L) and (U, R). A general characterization of the distributions on actions induced by OGCEs is provided in Section 5.

Our second example is based on the game (taken from Aumann (1974)) depicted in Fig. 3. This is a three-player game in which player 1 chooses the row, player 2 the column, and player 3 the matrix. A calculation shows that in any OCE of Γ_2 player 1 chooses D and player 2 chooses L. (Player 3 might choose A or B, depending on his private information.) In particular, each player gets a payoff of 1. By contrast, we now describe an OGCE of Γ_2 in which each player actually *receives* (ex post) a payoff of 3.

Figure 4a depicts a copy of the state space $\Omega = \{1, 2, 3, 4\}$, and player 1's possibility correspondence P^1. Figure 4b depicts a second copy of Ω, and player 2's possibility correspondence P^2. Player 3's possibility correspondence P^3 satisfies: $P^3(1) = P^3(2) = \{1, 2\}$, $P^3(3) = P^3(4) = \{3, 4\}$. The common prior π assigns probability $\frac{1}{4}$ to each state. Player 1 chooses U at every state of the world, player 2 chooses R at every state, and player 3

A

	L	R
U	0,8,0	3,3,3
D	1,1,1	0,0,0

B

	L	R
U	0,0,0	3,3,3
D	1,1,1	8,0,0

Γ_2

FIGURE 3

chooses A in states 1 and 2 and B in states 3 and 4. All this constitutes an OGCE of Γ_2 in which each player receives a payoff of 3. The possibility correspondences P^1 and P^2 satisfy nondelusion and KTYK (hence balancedness), but are not positively balanced.

Let us now consider two possible interpretations of a GCE. The first is

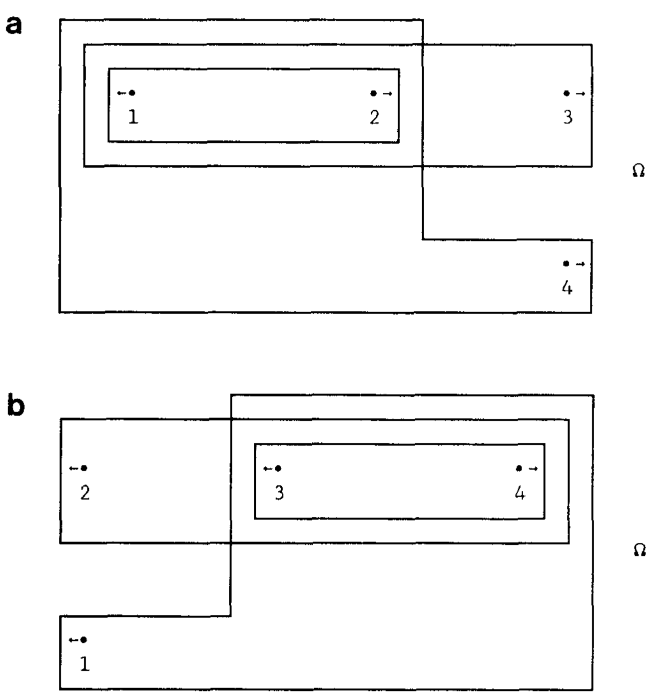

FIGURE 4

simply the counterpart to the conventional "common knowledge" view of equilibrium. Under this interpretation all aspects of the structure of the game, including the players' possibility correspondences, are common knowledge. In particular, it is common knowledge that the players make information processing errors—but there is nothing that they can do about it! Indeed it can be to the players' mutual advantage that they make such errors: witness the fact that in the OGCE of Γ_2 just described, the players receive (ex post) payoffs of 3, as opposed to 1 in any OCE of Γ_2. This first interpretation is not, however, one that we find compelling and so we now turn to an alternative interpretation.

The second interpretation assumes much less than the common knowledge hypothesized above. We think it more appropriate in the context of this paper to suppose that, far from the possibility correspondences being common knowledge, a player is not even aware of his own possibility correspondence. To see why this is more natural in the present context, consider the GCE of Matching Pennies described above. When $\omega = 2$ say, player 1 considers either $\omega = 1$ or $\omega = 2$ possible. Note that player 1 does *not* argue that since he was not informed of the set $\{1\}$, which he would have been had the true state been $\omega = 1$, but rather was informed of $\{1, 2\}$, only $\omega = 2$ is possible. (This latter line of reasoning would of course, if adopted, lead back to the supposition that player 1 processes information in accordance with a partition.) For player 1 to reason in this (counterfactual) fashion he would need to be aware of his own possibility correspondence. By contrast, the point of view of this paper is precisely that players may lack such complete self-awareness and that information processing errors are therefore to be expected. Indeed, it seems to us that people often take information at face value, and do not contemplate what other information they could have received. For example, how often when reading a newspaper does one go through the process of imagining how the article would have been written in different states of the world (even though such a process may well lead to a better understanding of the true state of affairs)?

Having discussed the "lack of self-awareness" alternative to the "common knowledge" hypothesis, we now turn to the second interpretation of a GCE. Players are assumed to make information processing errors but are unaware that they do so. At the same time, our definition of equilibrium presupposes that each player's expectations of how the other players move as a function of the state of the world are correct. (If one adopts the view of correlated equilibrium proposed in Aumann (1987), then a state of the world specifies each player's choice. In this case it is tautologically true that expectations of moves as a function of the state are correct.) Under our second interpretation, then, the notion of GCE is plausible insofar as the conjunction of the two hypotheses of "lack of self-awareness" and correct expectations concerning others' moves makes sense. We find the conjunction of these two hypotheses no less reasonable than is a model of information processing errors in the single-person case. There also, a decision maker is allowed to make mistakes in processing information but is assumed to assess correctly the consequences of any given state. Likewise, in the present context, a player may make information processing errors but is assumed to assess correctly the consequences of any given state, namely the moves of the other players at that state.

In the definition of a GCE players are permitted to make information processing errors about the state of the world. On the other hand, they are not allowed to be mistaken about the actions chosen by the other players as a function of the state of the world. Note, however, that players *can* make mistakes, in an indirect fashion, about other player's actions by making mistakes about ω. In the GCE of Matching Pennies described above, when $\omega = 2$ player 1 "should" recognize that since he has not been informed of the set $\{1\}$ but has been informed of $\{1, 2\}$ the state must be $\omega = 2$. In other words, he should deduce that player 2 is playing R. Instead, player 1 acts as if he ignores this finer information and places probability $\frac{4}{7}$ on player 2 playing L. Given the fact that he is playing U at $\omega = 2$, we might say that player 1 ignores the "bad" news that player 2 is actually playing R. Observe that what is "good" or "bad" news for player 1 is determined endogenously by the equilibrium.

4. Decision-Theoretic Equivalence

In this section we demonstrate an equivalence between correlated equilibria that allow for information processing errors and correlated equilibria in which the agents do not make such errors but may have different priors. This result is based on the following notion of decision-theoretic equivalence between equilibria of a game.

Definition 4.1. For a fixed game Γ, let $\langle \Omega ; \pi^i, P^i, f^i \rangle$ and $\langle \tilde{\Omega} ; \tilde{\pi}^i, \tilde{P}^i, \tilde{f}^i \rangle$ be SGCEs. The two equilibria are *decision-theoretically equivalent* if for every i there is an isomorphism $\phi^i \colon \tilde{\mathbb{P}}^i \to \mathbb{P}^i$ such that:

(1) $\tilde{f}^i(\tilde{\omega}) = f^i(\omega)$ when $P^i(\omega) = \phi^i(\tilde{P}^i(\tilde{\omega}))$;

(2) for $\tilde{R}^i \in \tilde{\mathbb{P}}^i$ and $R^i = \phi^i(\tilde{R}^i)$

$$\sum_{\tilde{\omega} \in \tilde{R}^i} \tilde{\pi}^i(\tilde{\omega} | \tilde{R}^i) u^i(a^i, \tilde{f}^{-i}(\tilde{\omega})) = \sum_{\omega \in R^i} \pi^i(\omega | R^i) u^i(a^i, f^{-i}(\omega))$$

for all $a^i \in A^i$.

It is easy to see that this notion of decision-theoretic equivalence is indeed an equivalence relation. If one SGCE is decision-theoretically equivalent to another, then behaviorally the two equilibria are equivalent in the sense that strategies and conditional expected payoffs agree.

PROPOSITION 4.1. *Let $\langle \Omega; \pi^i, P^i, f^i \rangle$ be an SGCE of a game Γ. Then there is a decision-theoretically equivalent SCE $\langle \tilde{\Omega}; \tilde{\pi}^i, H^i, \tilde{f}^i \rangle$ of* Γ.

Proof. Let $\tilde{\Omega} = \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$ and for each i let $H^i(R^1, \ldots, R^n) = \{R^i\} \times \mathbb{P}^{-i}$. By construction, $\mathbb{H}^i$ is a partition and there is an isomorphism ϕ^i: $\mathbb{H}^i \to \mathbb{P}^i$. Let $\tilde{\pi}^i$ be defined by

$$\tilde{\pi}^i(R^1, \ldots, R^n | \{R^i\} \times \mathbb{P}^{-i}) = \pi^i(\{\omega : P^j(\omega) = R^j \text{ for } j \neq i\} | R^i)$$

and

$$\tilde{\pi}^i(\{R^i\} \times \mathbb{P}^{-i}) = \frac{1}{\# \mathbb{P}^i}.$$

The probability measure $\tilde{\pi}^i$ is defined immediately from the above conditional and marginal. Define $\tilde{f}^i$: $\tilde{\Omega} \to A^i$ by $\tilde{f}^i(R^1, \ldots, R^n) = f^i(\omega)$ for ω such that $P^i(\omega) = R^i$. Now, using the definitions, it follows that for all $R^i \in \mathbb{P}^i$ and $a^i \in A^i$

$$\sum_{R^{-i} \in \mathbb{P}^{-i}} \tilde{\pi}^i(R^1, \ldots, R^n | \{R^i\} \times \mathbb{P}^{-i}) u^i(a^i, \tilde{f}^{-i}(R^1, \ldots, R^n))$$

$$= \sum_{R^{-i} \in \mathbb{P}^{-i}} \pi^i(\{\omega : P^j(\omega) = R^j \text{ for } j \neq i\} | R^i) u^i(a^i, \tilde{f}^{-i}(R^1, \ldots, R^n))$$

$$= \sum_{R^{-i} \in \mathbb{P}^{-i}} \sum_{\{\omega : P^j(\omega) = R^j \text{ for } j \neq i\}} \pi^i(\omega | R^i) u^i(a^i, f^{-i}(\omega))$$

$$= \sum_{\omega \in R^i} \pi^i(\omega | R^i) u^i(a^i, f^{-i}(\omega)). \quad \blacksquare$$

Proposition 4.1 says that the notion of an SGCE is not decision-theoretically more general than that of an SCE. However, OGCEs *are* more general than OCEs. That is, there are GCEs in which the players have a common prior, such that in any equivalent CE the players are required to have different priors. For example, refer back to the OGCE of Matching Pennies described in Section 3. The state space $\tilde{\Omega}$ and the players' conditional probability distributions in the decision-theoretically equivalent CE constructed according to the proof of Proposition 4.1 are illustrated in Fig. 5.

Could these conditionals have arisen from a common prior on $\tilde{\Omega}$? The answer is no, as can be seen either by direct calculation (by showing that

P^1 \ P^2	{1,4,5}	{2,3,6}
{1}	1	0
{1,2}	4/7	3/7
{1,3}	4/7	3/7
{4,6}	3/7	4/7
{5,6}	3/7	4/7
{6}	0	1

$\tilde{\Omega}$

1's conditionals

P^1 \ P^2	{1,4,5}	{2,3,6}
{1}	2/5	0
{1,2}	0	3/10
{1,3}	0	3/10
{4,6}	3/10	0
{5,6}	3/10	0
{6}	0	2/5

$\tilde{\Omega}$

2's conditionals

FIGURE 5

the restrictions that such a prior would have to satisfy are inconsistent), or by recalling that in any OCE of Matching Pennies the conditional expected payoffs to the players are always 0 (Aumann, 1974). Hence Proposition 4.1 implies that, starting from an equilibrium in which players have a common prior but may make information processing errors, there is a decision-theoretically equivalent equilibrium in which players have partitions but may have different priors.

We now establish a converse to Proposition 4.1: we show that any SCE is decision-theoretically equivalent to an OGCE. Hence, in the context of correlated equilibrium, arbitrary differences in players' priors can be interpreted as having arisen from a situation in which the players have a common prior but make information processing errors.

PROPOSITION 4.2. *Let $\langle \Omega ; \pi^i, H^i, f^i \rangle$ be an SCE of a game Γ. Then there is a decision-theoretically equivalent OGCE $\langle \tilde{\Omega} ; \tilde{\pi}, \tilde{P}^i, \tilde{f}^i \rangle$ of Γ in which the $\tilde{P}^i$'s satisfy KTYK.*

Proof. Let $\tilde{\Omega} = \Omega \times \{1, \ldots, n\}$. Player i's possibility correspondence $\tilde{P}^i$ is defined by

$$\tilde{P}^i(\omega, j) = \{(\omega', i) \in \tilde{\Omega} : \omega' \in H^i(\omega)\}$$

for $(\omega, j) \in \tilde{\Omega}$. Let

$$\pi^*(\omega, i) = \pi^i(\omega | P^i(\omega))$$

for $(\omega, i) \in \tilde{\Omega}$ and

$$\tilde{\pi}(\omega, i) = \frac{\pi^*(\omega, i)}{\sum_{(\omega', j) \in \tilde{\Omega}} \pi^*(\omega', j)}.$$

Finally, the map $\tilde{f}^i$: $\tilde{\Omega} \rightarrow A^i$ is given by $\tilde{f}^i(\omega, j) = f^i(\omega)$ for $(\omega, j) \in \tilde{\Omega}$. ■

In the decision-theoretically equivalent OGCE just constructed, the possibility correspondences satisfy KTYK but they do not satisfy nondelusion. (Also, the players know their own actions—see Section 5.) Imposing nondelusion *is* restrictive. For example, in Matching Pennies there is an SCE in which all the conditional expected payoffs are 1. This clearly cannot happen in an OGCE of Matching Pennies if nondelusion is satisfied.

Taken together, Propositions 4.1 and 4.2 provide an explanation of differences in priors in terms of bounded rationality on the part of the players. The standard assumption in game theory has been what Aumann has termed the "Harsanyi doctrine," namely that all players begin with a common prior. In this case it is impossible for rational players to agree to bet or trade risky securities with one another based solely on differences in information, when that information is represented by partitions (Milgrom and Stokey, 1982); Geanakoplos and Sebenius, 1983). If the possibility of arbitrary differences in priors between players is admitted, then there is of course no difficulty in explaining betting and securities trading. But the approach of postulating at the outset that priors may disagree has proved rather unpopular—only a small minority of papers (e.g., Harrison and Kreps, 1978) consider "subjective" priors. We have seen that differences in priors can be justified as a manifestation of bounded rationality on the part of the players. This suggests that speculative behavior might usefully be explored from this point of view. In fact, the OGCE of Matching Pennies described in Section 3 shows that betting can occur with a common prior and information processing errors: the players are effectively betting with each other over which outcome of the game will obtain. Geanakoplos (1989) characterizes the kinds of information processing errors that permit speculation.

Remark 4.1. Having allowed for information processing errors, it is natural to allow for errors in calculating conditional probabilities as well. Nevertheless, Proposition 4.1 shows that information processing errors subsume errors of the latter kind. To see this, consider a further generalization of correlated equilibrium in which each player i has a belief function δ^i: $\Omega \rightarrow \Delta(\Omega)$ giving i's beliefs at each state ω (cf. the discussion in Section 2). An equilibrium of a game Γ would then be a collection $\langle \Omega ; \delta^i, f^i \rangle$ where for every i and each $\omega \in \Omega$:

(1) $\delta^i(\omega') = \delta^i(\omega)$ implies $f^i(\omega') = f^i(\omega)$;

$$(2) \sum_{\omega' \in P^i(\omega)} \delta^i(\omega)(\omega')u^i(f^i(\omega), f^{-i}(\omega')) \geq \sum_{\omega' \in P^i(\omega)} \delta^i(\omega)(\omega')u^i(a^i, f^{-i}(\omega'))$$

for all $a^i \in A^i$. Let $D^i \subset \Delta(\Omega)$ denote the range of δ^i. By analogy with Definition 4.1, we say that two equilibria $\langle \Omega ; \delta^i, f^i \rangle$ and $\langle \tilde{\Omega} ; \tilde{\delta}^i, \tilde{f}^i \rangle$ of Γ are decision-theoretically equivalent if for every i there is an isomorphism ϕ^i: $\tilde{D}^i \rightarrow D^i$ such that whenever $\delta^i(\omega) = \phi^i(\tilde{\delta}^i(\tilde{\omega}))$ then $\tilde{f}^i(\tilde{\omega}) = f^i(\omega)$ and the conditional expected payoffs to i at $\tilde{\omega}$ and ω are equal. A careful reading of the proof of Proposition 4.1 shows that any equilibrium $\langle \Omega ; \delta^i, f^i \rangle$ of Γ is decision-theoretically equivalent to an SCE (and hence to an SGCE) of Γ. Thus no new correlated equilibria arise by miscalculation of conditional probabilities.

We close this section by mentioning briefly the connection between the results of this section and the solution concept of rationalizability due to Bernheim (1984) and Pearce (1984). In a two-person game Γ, the set of conditional expected payoffs to a player i from the SCEs of Γ coincides with the set of i's rationalizable payoffs in Γ (Brandenburger and Dekel, 1987, Proposition 2.1). The same equivalence holds in games with more than two players, provided the term "rationalizable" is replaced with "correlated rationalizable" (Brandenburger and Dekel, p. 1394). Hence Proposition 4.1 implies that conditional expected payoffs from GCEs are (correlated) rationalizable payoffs.

5. Characterization of Equilibrium Distributions

This section characterizes distributions on actions that arise from OGCEs. Recall that for an OCE $\langle \Omega ; \pi, H^i, f^i \rangle$ of Γ, there is a naturally induced distribution on actions $\lambda \in \Delta(A^1 \times \cdots \times A^n)$ given by

$$\lambda(a^1, \ldots, a^n) = \pi(\{\omega : f^i(\omega) = a^i, i = 1, \ldots, n\})$$

for $(a^1, \ldots, a^n) \in A^1 \times \cdots \times A^n$ (see Aumann, 1987). This will be called an *objective correlated equilibrium distribution* (OCED). There is a well-known characterization of the set of all OCEDs. Given a probability measure $\lambda \in \Delta(A^1 \times \cdots \times A^n)$, and an $a^i \in A^i$ such that $\lambda(\{a^i\} \times A^{-i}) > 0$, let $\lambda(\cdot|a^i) \in \Delta(A^{-i})$ be the conditional probability measure on the actions of the other players. Given a game Γ, a distribution $\lambda \in \Delta(A^1 \times \cdots \times A^n)$ is an OCED if and only if for every i and each $a^i \in A^i$ with $\lambda(\{a^i\} \times A^{-i}) > 0$

$$\sum_{a^{-i} \in A^{-i}} \lambda(a^{-i}|a^i)u^i(a^i, a^{-i}) \geq \sum_{a^{-i} \in A^{-i}} \lambda(a^{-i}|a^i)u^i(b^i, a^{-i})$$

for all $b^i \in A^i$. The set of all OCEDs is thus a closed, convex set defined by the above system of linear inequalities.

Given an OGCE, there is a precisely analogous induced distribution on actions, to be called an *objective generalized correlated equilibrium distribution* (OGCED). To illustrate these definitions, Fig. 6 depicts first the (unique) OCED of Matching Pennies, and then the OGCED induced by the OGCE of Matching Pennies described in Section 3.

In characterizing OGCEDs in general, we will make use of the following assumption. A player i is said to *know his own actions* if for each $R^i \in \mathbb{P}^i$ there is an $a^i \in A^i$ such that $f^i(\omega) = a^i$ for all $\omega \in R^i$. This assumption says that i is sure about what he is playing. (However, i may be mistaken if nondelusion is violated.) Note that a player always knows his own actions (correctly) if his information is described by a partition.

We make two different sets of assumptions in characterizing OGCEDs. In Proposition 5.1 we suppose that the players know their own actions and that the possibility correspondences satisfy nondelusion and either balancedness or KTYK. The proposition provides a way of calculating whether a distribution on actions is an OGCED. In Proposition 5.2 we suppose the players know their own actions and that the possibility correspondences satisfy nondelusion and positive balancedness—in this case all OGCEDs are OCEDs.

In order to state the first result, we need some notation. Given a distribution $\lambda \in \Delta(A^1 \times \cdots \times A^n)$ and an $a^i \in A^1$ such that $\lambda(\{a^i\} \times A^{-i}) > 0$, let

$$Q_\lambda(a^i) = \Big\{ q \in \Delta(A^{-i}) : \text{Supp } q \subset \text{Supp } \lambda(\cdot | a^i), \sum_{a^{-i} \in A^{-i}} q(a^{-i})[u^i(a^i, a^{-i}) - u^i(b^i, a^{-i})] \geq 0 \quad \forall b^i \in A^i \Big\},$$

where Supp denotes the support of a measure. In words, $Q_\lambda(a^i)$ is the set of all distributions q on A^{-i}, with support contained in that of $\lambda(\cdot | a^i)$,

	L	R
U	$\frac{1}{4}$	$\frac{1}{4}$
D	$\frac{1}{4}$	$\frac{1}{4}$

OCED λ

	L	R
U	$\frac{1}{5}$	$\frac{3}{10}$
D	$\frac{3}{10}$	$\frac{1}{5}$

OGCED λ

FIGURE 6

under which a^i is an optimal action for i. Note that $Q_\lambda(a^i)$ is a compact, convex subset of $\Delta(A^{-i})$. Given a set Y, let aff Y denote the affine hull of Y. That is, aff $Y = \{\Sigma_m \alpha_m y_m : y_m \in Y \text{ and } \Sigma_m \alpha_m = 1\}$.

PROPOSITION 5.1. *Given a game Γ, a distribution $\lambda \in \Delta(A^1 \times \cdots \times A^n)$ is an OGCED induced by an OGCE in which the players know their own actions and the possibility correspondences satisfy nondelusion and balancedness if and only if for every i and each $a^i \in A^i$ with $\lambda(\{a^i\} \times A^{-i}) > 0$, $\lambda(\cdot|a^i) \in$ aff $Q_\lambda(a^i)$.*[3]

Remark 5.1. The stronger requirement that $\lambda(\cdot|a^i) \in Q_\lambda(a^i)$ is exactly the condition for λ to be an OCED.

Remark 5.2. As the proof will make clear, Proposition 5.1 remains true if the assumption that the possibility correspondences satisfy balancedness is replaced by the assumption that they satisfy KTYK. In general, under the hypothesis of nondelusion, KTYK is a more restrictive assumption than balancedness. By contrast, in the context of OGCEs in which the players know their own actions and the possibility correspondences satisfy nondelusion, KTYK and balancedness turn out to be equivalent.

Remark 5.3. Proposition 5.1 implies that if $\lambda(\{a^i\} \times A^{-i}) > 0$, then a^i is a correlated rationalizable action for player i. To see why, for each player i let $B^i = \{a^i \in A^i: \lambda(\{a^i\} \times A^{-i}) > 0\}$. For any $a^i \in B^i$, $Q_\lambda(a^i) \neq \emptyset$ and for any $q \in Q_\lambda(a^i)$, Supp $q \subset B^{-i}$. Hence there is a subset $B^1 \times \cdots \times B^n \subset A^1 \times \cdots \times A^n$ such that for each i, every $a^i \in B^i$ is a best reply to a distribution on B^{-i}. That is, a^i is correlated rationalizable.

COROLLARY 5.1. *Given a game Γ, the set of all OGCEDs induced by OGCEs in which the players know their own actions and the possibility correspondences satisfy nondelusion and either balancedness or KTYK is nonempty and convex, but may not be closed.*

Before providing proofs, we illustrate Proposition 5.1 and Corollary 5.1 in the context of Matching Pennies. Recall that the unique OCED for Matching Pennies assigns probability $\frac{1}{4}$ to each pair of actions. Proposition 5.1 implies that the set of OGCEDs induced by OGCEs in which the players know their own actions and the possibility correspondences satisfy nondelusion and balancedness is much larger: it consists of all strictly positive distributions on $\{U, D\} \times \{L, R\}$. Note that this is a convex set, but is *not* closed. To see that any strictly positive λ is an OGCED with the aforementioned properties, observe that for player 1, U is a best reply to

[3] At the time we were working on the first draft of this paper, Dov Samet mentioned to us that he was also working toward a result similar to our Proposition 5.1.

the distributions (1, 0) and $(\frac{1}{2}, \frac{1}{2})$ on $\{L, R\}$. Hence if λ has full support, aff $Q_\lambda(U) = \Delta(\{L, R\})$ and Proposition 5.1 places no restriction on $\lambda(\cdot|U)$. The argument for D, L, and R is analogous. Conversely, no OGCED λ with the aforementioned properties can assign probability 0 to any pair of actions. Suppose $\lambda(U, L) = 0$. Then if $\lambda(U, R) > 0$, $Q_\lambda(U) = \varnothing$ hence λ cannot be an OGCED. Thus $\lambda(U, R) = 0$, but then by symmetry λ is identically 0, which is a contradiction.

Proof of Proposition 5.1. To prove sufficiency, we construct an OGCE by subdividing the elements of $A^1 \times \cdots \times A^n$ into states $\omega \in \Omega$. The construction proceeds player-by-player and for each player, action-by-action. So fix a player i and an $a^i \in A^i$ with $\lambda(\{a^i\} \times A^{-i}) > 0$. By hypothesis, $\lambda(\cdot|a^i) = \sum_m \alpha_m q_m$ for $q_m \in Q_\lambda(a^i)$ and $\sum_m \alpha_m = 1$. In fact, since $Q_\lambda(a^i)$ is convex, we can write $\lambda(\cdot|a^i) = \alpha q + (1 - \alpha)q'$ for $q, q' \in Q_\lambda(a^i)$. Without loss of generality $\alpha < 1$. Note that if $0 < \beta < 1$ and β is sufficiently close to 1, then $\beta q + (1 - \beta)\lambda(\cdot|a^i) \in Q_\lambda(a^i)$.

Since Supp $q \subset$ Supp $\lambda(\cdot|a^i)$, we can find a section S_0 of the rectangle $\{a^i\} \times A^{-i}$ such that $\lambda(\cdot|S_0) = q$. Letting $\sim S_0$ be the complement of S_0 in $\{a^i\} \times A^{-i}$ and $\tilde{q} = \lambda(\cdot|\sim S_0)$, we know that $\lambda(\cdot|a^i)$ lies on the line segment from q through $\tilde{q}$. Hence if $0 < \gamma < 1$ and γ is sufficiently close to 1, $\gamma q + (1 - \gamma)\tilde{q} \in Q_\lambda(a^i)$. Now divide up $\sim S_0$ into disjoint sections $S_1, \ldots, S_K$ such that $\lambda(\cdot|S_k) = \tilde{q}$ for $k = 1, \ldots, K$. Let $T_k = S_0 \cup S_k$ for $k = 1, \ldots, K$. Then if K is sufficiently large, $\lambda(\cdot|T_k) \in Q_\lambda(a^i)$ for all k.

Player i's possibility correspondence P^i is given by

$$P^i(\omega) = \begin{cases} S_0 & \text{if } \omega \in S_0; \\ T_k & \text{if } \omega \in S_k \text{ for } k = 1, \ldots, K. \end{cases}$$

Clearly, nondelusion and KTYK are satisfied and hence balancedness also holds. (In verifying balancedness directly, the nontrivial self-evident sets to check are of the form $\{a^i\} \times A^{-i}$. The balancing weights are 1 for each T_k, $k = 1, \ldots, K$, and $-(K - 1)$ for S_0.) By construction, player i knows his own actions.

We divide up any other rectangle $\{b^i\} \times A^{-i}$, $b^i \neq a^i$, in a similar fashion. The same procedure is then repeated for every other player. At the end, the states $\omega \in \Omega$ consist of the intersections of all the divisions of rectangles.

To prove necessity, let $\langle\Omega; \pi, P^i, f^i\rangle$ be an OGCE of Γ in which the players know their own actions and the possibility correspondences satisfy nondelusion and balancedness. The first step is to show that if P^i is balanced, then for any self-evident set $E \subset \Omega$ and any $F \subset \Omega$

$$\pi(F|E) \in \text{aff}\, \{\pi(F|R^i) \colon R^i \in \mathbb{P}^i, R^i \subset E\}.$$

To see this, write

$$\pi(F|E) = \frac{1}{\pi(E)} \sum_{\omega \in \Omega} \chi_E(\omega)\chi_F(\omega)\pi(\omega)$$

$$= \frac{1}{\pi(E)} \sum_{\omega \in \Omega} \sum_{\substack{R^i \in \mathbb{P}^i \\ R^i \subset E}} \beta(R^i)\chi_{R^i}(\omega)\chi_F(\omega)\pi(\omega)$$

using balancedness. Hence

$$\pi(F|E) = \frac{1}{\pi(E)} \sum_{\substack{R^i \in \mathbb{P}^i \\ R^i \subset E}} \beta(R^i)\pi(R^i) \frac{1}{\pi(R^i)} \sum_{\omega \in \Omega} \chi_{R^i}(\omega)\chi_F(\omega)\pi(\omega)$$

$$= \frac{1}{\pi(E)} \sum_{\substack{R^i \in \mathbb{P}^i \\ R^i \subset E}} \beta(R^i)\pi(R^i)\pi(F|R^i).$$

But using balancedness it is easy to show that

$$\frac{1}{\pi(E)} \sum_{\substack{R^i \in \mathbb{P}^i \\ R^i \subset E}} \beta(R^i)\pi(R^i) = 1$$

and so

$$\pi(F|E) \in \text{aff}\,\{\pi(F|R^i)\colon R^i \in \mathbb{P}^i,\ R^i \subset E\}.$$

Now for any i and $a^i \in A^i$, let $E(a^i) = \{\omega \in \Omega\colon f^i(\omega) = a^i\}$. Since i knows his own actions and P^i satisfies nondelusion, $E(a^i)$ is self-evident. Hence, letting $E(a^{-i}) = \{\omega \in \Omega\colon f^{-i}(\omega) = a^{-i}\}$,

$$\lambda(a^{-i}|a^i) = \pi(E(a^{-i})|E(a^i)) \in \text{aff}\,\{\pi(E(a^{-i})|R^i)\colon R^i \in \mathbb{P}^i,\ R^i \subset E(a^i)\}.$$

But for any $R^i \in \mathbb{P}^i$ with $R^i \subset E(a^i)$, the measure $q \in \Delta(A^{-i})$ given by $q(a^{-i}) = \pi(E(a^{-i})|R^i)$ for $a^{-i} \in A^{-i}$ is a member of $Q_\lambda(a^i)$. (The optimality of a^i given q follows from the hypothesis that $\langle\Omega; \pi, P^i, f^i\rangle$ is an OGCE, and the support condition is straightforward to verify.) Thus $\lambda(\cdot|a^i) \in$ aff $Q_\lambda(a^i)$. ■

Proof of Corollary 5.1. The set of all OGCEDs is nonempty since any OCED λ is also an OGCED. To show convexity, suppose λ, $\tilde{\lambda}$ are OGCEDs and let $\mu = \alpha\lambda + (1 - \alpha)\tilde{\lambda}$ for $0 < \alpha < 1$. We have to show that if $\mu(\{a^i) \times A^{-i}) > 0$, then $\mu(\cdot|a^i) \in$ aff $Q_\mu(a^i)$. Now $\mu(\cdot|a^i) = \beta\lambda(\cdot|a^i) + (1 - \beta)\tilde{\lambda}(\cdot|a^i)$ for some $0 \leq \beta \leq 1$. (If $\lambda(\{a^i\} \times A^{-i}) = 0$ then $\beta = 0$. If

$\tilde{\lambda}(\{a^i\} = 0$ then $\beta = 1$.) Also $Q_\lambda(a^i) \subset Q_\mu(a^i)$ so $\lambda(\cdot|a^i) \in \text{aff } Q_\mu(a^i)$. Similarly, $\tilde{\lambda}(\cdot|a^i) \in \text{aff } Q_\mu(a^i)$. Hence $\mu(\cdot|a^i) \in \text{aff } Q_\mu(a^i)$ since aff $Q_\mu(a^i)$ is convex. ∎

Our final result shows that strengthening the hypothesis of balancedness in Proposition 5.1 to that of positive balancedness leads to an equivalence between OGCEDs and OCEDs. Thus in the context of OGCEs in which the players know their own actions and the possibility correspondences satisfy nondelusion, positive balancedness is no more general than assuming a partition. Proposition 5.2 mirrors the Generalized Sure Thing Principle established for single-person decision problems and Nash equilibria in Geanakoplos (1989).

PROPOSITION 5.2. *Let $\langle \Omega; \pi, P^i, f^i \rangle$ be an OGCE of a game Γ in which the players know their own actions and the possibility correspondences satisfy nondelusion and positive balancedness. Then the induced OGCED λ is an OCED of Γ.*

Proof. Repeat the necessity part of the proof of Proposition 5.1, observing that because of positive balancedness the conclusion that $\lambda(\cdot|a^i) \in \text{aff } Q_\lambda(a^i)$ can be strengthened to assert that $\lambda(\cdot|a^i)$ lies in the *convex* hull of $Q_\lambda(a^i)$. Since $Q_\lambda(a^i)$ is convex, it follows (using Remark 5.1) that λ is an OCED. ∎

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