Supplementary Appendix for "Sequential Mechanisms for Evidence Acquisition"

Elchanan Ben-Porath
Eddie Dekel
Barton L. Lipman

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1 Proof of Lemma 2

For notational brevity, let

$$\pi(d) = \mathrm{E}_v \left[\sum_i P_i(v \mid d) v_i \right]$$

and

$$U_i(d) = \mathrm{E}_v P_i(v \mid d) - c_i e_i(d).$$

Because d is a probability measure over dynamic mechanisms, $\pi(d)$ and the $U_i(d)$'s are linear in d in the sense that

$$\pi(\alpha d + (1-\alpha)d') = \alpha\pi(d) + (1-\alpha)\pi(d'), \quad \forall \alpha \in [0,1],\ d, d' \in D.$$

Written this way, $\pi(d)$ and the $U_i(d)$'s are convex and concave functions of d. This is because convex combinations of mixed strategies induce convex combinations of the distributions over outcomes.

So our problem is

$$\max_{d \in D} \pi(d)$$

subject to

$$U_i(d) \geq 0, \quad \forall i.$$

Let D^* denote the set of d's solving this constrained optimization problem. Since the objective function is continuous and the feasible set nonempty and compact, we know $D^* \neq \emptyset$.

Let $D^{**}(\lambda)$ denote the set of d's solving

$$\max_{d \in D} \pi(d) + \sum_i \lambda_i U_i(d)$$

and let D^{**} denote the set of d's such that there exists $\lambda^* \in \mathbf{R}_+^N$ with $d \in \Delta^*(\lambda^*)$ such that (a) $U_i(d) \geq 0$ for all i and (b) $\lambda_i^* U_i(d) = 0$ for all i.

We now show that $D^* = D^{**}$.

First, we show $D^{**} \subseteq D^*$. Fix any $d^* \in D^{**}$ and let λ^* be the associated vector in $\mathbf{R}_+^N$. Suppose, contrary to our claim, that there exists $\hat{d}$ with $U_i(\hat{d}) \geq 0$ for all i and $\pi(\hat{d}) > \pi(d^*)$. By $d^* \in D^{**}(\lambda^*)$,

$$\pi(d^*) + \sum_{i=1}^N \lambda_i^* U_i(d^*) \geq \pi(\hat{d}) + \sum_{i=1}^N \lambda_i^* U_i(\hat{d}).$$

Because $\lambda_i^* U_i(d^*) = 0$ for all i, this implies

$$\pi(d^*) \geq \pi(\hat{d}) + \sum_{i=1}^N \lambda_i^* U_i(\hat{d}).$$

Because $\lambda_i^* \geq 0$ for all i and $U_i(\hat{d}) \geq 0$ for all i, this implies $\pi(d^*) \geq \pi(\hat{d})$, a contradiction. Hence $D^{**} \subseteq D^*$.

The proof of the converse is a simplification of the proof of Theorem 1, Section 8.3, of Luenberger (1969). Fix any d^* in D^*. Let

$$A = \{u = (u_0, u_1, \ldots, u_N) \in \mathbf{R}^{N+1} \mid \exists d \in D \text{ with } u_0 \leq \pi(d) \text{ and } u_i \leq U_i(d),\ \forall i = 1, \ldots, N\}$$

$$B = \{u = (u_0, u_1, \ldots, u_N) \in \mathbf{R}^{N+1} \mid u_0 \geq \pi(d^*) \text{ and } u_i \geq 0,\ \forall i = 1, \ldots, N\}.$$

Obviously, both sets are nonempty as $(\pi(d^*), 0, 0, \ldots, 0)$ is in both sets.

Also, both sets are convex. The proof for B is trivial. For A, suppose u and u' are elements of A and fix any $\alpha \in (0, 1)$. Since $u \in A$, there exists $d \in D$ satisfying

$$u_0 \leq \pi(d)$$

$$u_i \leq U_i(d),\ \forall i$$

and let $d' \in \Delta$ satisfy the analog for u'. Then we have

$$\alpha u_0 + (1-\alpha) u_0' \leq \alpha \pi(d) + (1-\alpha)\pi(d') = \pi(\alpha d + (1-\alpha) d')$$

and

$$\alpha u_i + (1-\alpha)u_i' \leq \alpha U_i(d) + (1-\alpha)U_i(d') = U_i(\alpha d + (1-\alpha)d'),$$

implying $\alpha u + (1-\alpha)u' \in A$.

Also, we have $A \cap \text{int}(B) = \emptyset$. To see this, suppose to the contrary that there is $u \in \text{int}(B)$ with $u \in A$. Because $u \in \text{int}(B)$, we have $u_0 > \pi(d^*)$ and $u_i > 0$ for all i. Because $u \in A$, there exists $d \in D$ with $\pi(d) \geq u_0 > \pi(d^*)$ and $U_i(d) \geq u_i > 0$ for all i. But this contradicts $d^* \in D^*$ as d satisfies the constraints and gives a higher payoff than d^*.

By the Separating Hyperplane Theorem, there exists $p \in \mathbf{R}^{N+1}$, $p \neq 0$, such that

$$p_0 u_0 + \sum_{i=1}^{N} p_i u_i \leq p_0 \hat{u}_0 + \sum_{i=1}^{N} p_i \hat{u}_i, \quad \forall u \in A, \ \hat{u} \in B.$$

We now show that $p_i \geq 0$ for all i. Suppose to the contrary that some $p_i < 0$. Given the definition of B, we could make the corresponding component of $\hat{u}$ arbitrarily large and violate this inequality, a contradiction.

Also, $p_0 > 0$. To see this, suppose that $p_0 = 0$. We know that $(\pi(d^*), 0, \ldots, 0) \in B$, so this implies

$$\sum_{i=1}^{N} p_i u_i \leq 0,$$

for all $u \in A$. But consider the $d \in D$ where we randomize uniformly over which agent to ask first and always give her the good. For this procedure, $U_i(d) = (1 - c_i)/N > 0$ for all i. Hence there exists $u \in A$ with $u_i > 0$ for $i = 1, \ldots, N$. Hence the only way this inequality could hold is if $p_i = 0$ for all i. But we know $p \neq 0$, a contradiction.

For $i = 1, \ldots, N$, let $\lambda_i = p_i/p_0$. Then we have $\lambda \in \mathbf{R}^N_+$ with

$$u_0 + \sum_{i=1}^{N} \lambda_i u_i \leq \hat{u}_0 + \sum_{i=1}^{N} \lambda_i \hat{u}_i, \quad \forall u \in A, \ \hat{u} \in B.$$

Again, $(\pi(d^*), 0, \ldots, 0) \in B$, so this implies

$$\pi(d^*) \geq u_0 + \sum_{i=1}^{N} \lambda_i u_i, \quad \forall u \in A.$$

For every $d \in D$, $(\pi(d), U_1(d), \ldots, U_N(d)) \in A$, so this implies

$$\pi(d^*) \geq \pi(d) + \sum_{i=1}^{N} \lambda_i U_i(d), \quad \forall d \in D.$$

In particular, $d^* \in D$, so this implies

$$\pi(d^*) \geq \pi(d^*) + \sum_i \lambda_i U_i(d^*).$$

Because $\lambda_i \geq 0$ for all i and $U_i(d^*) \geq 0$ for all i, we have $\lambda_i U_i(d^*) = 0$ for all i. Hence

$$\pi(d^*) = \max_{d \in D} \left[\pi(d) + \sum_{i=1}^{N} \lambda_i U_i(d) \right].$$

Rephrasing, this shows that there exists $\lambda \in \mathbf{R}_+^N$ with $d^* \in D^{**}(\lambda)$ with $U_i(d^*) \geq 0$ and $\lambda_i U_i(d^*) = 0$ for all i. Hence $d^* \in D^{**}$, completing the proof. ▌

2 Border

In this section, we state and prove a version of a result in Border (1991). Lemma 1 below is essentially Border's Lemma 5.1 and Theorem 1 is essentially his Lemma 6.1.

First, we introduce some notation and terminology. In this section only, we denote the set of types for agent i by T_i and assume T_i is finite and not a singleton for each i. We consider *allocations* $P = (P_1, \ldots, P_N)$ with $P_i : T \to [0, 1]$ with $\sum_i P_i(t) \leq 1$ for all $t \in T$. Given P, we let $p = (p_1, \ldots, p_N)$ denote the *interim probabilities* where

$$p_i(t_i) = \sum_{t_{-i} \in T_{-i}} \mu_{-i}(t_{-i}) P_i(t_i, t_{-i}),$$

where $\mu_j(t_j)$ is the prior over T_j and we assume type distributions are independent across agents. When p and P are related in this fashion, we say P *generates* p.

Lemma 1. *Any interim allocation p satisfies the following for every $(\hat{T}_1, \ldots, \hat{T}_N)$ with $\hat{T}_i \subseteq T_i$ for all i:*

$$\sum_i \sum_{t_i \in \hat{T}_i} p_i(t_i) \mu_i(t_i) \leq 1 - \prod_i \left[1 - \mu_i(T_i)\right].$$

Proof. The left–hand side is the probability that the good is allocated to some type in $\cup_i \hat{T}_i$. The right–hand side is the probability that at least one agent's type is in her $\hat{T}_i$ set. ▌

A *hierarchical allocation* is an allocation P that can be constructed as follows. We have a ranking function R which maps $\cup_i T_i$ to $\{1, \ldots, K\}$ for some positive integer K.

We assume that for every $k < K$, there is exactly one i such that $R(t_i) = k$ for some $t_i \in T_i$. Note that this restriction does *not* apply to rank K — there may be no or many agents with types at rank K.

Then given a type profile $t = (t_1, \ldots, t_N)$, either all agents have rank K or there is a unique i with $R(t_i) < R(t_j)$ for all $j \neq i$. If all agents have rank K, then $P_j(t) = 0$ for all j. If there is a unique i with $R(t_i) < R(t_j)$ for all $j \neq i$, then $P_i(t) = 1$. In other words, unless all agents are in the lowest rank, the agent who has the highest ranked type receives the good (where higher ranks have lower numbers).

We say that p is a *hierarchical interim probability* if it is generated by a hierarchical allocation P. Of course, the collection of hierarchical interim probabilities is a subset of the interim probabilities.

Theorem 1. *The set of hierarchical interim probabilities is the set of extreme points of the set of interim probabilities. That is, a function p is an interim probability if and only if it is a convex combination of hierarchical interim probabilities.*

Proof. We first show that any hierarchical interim probability p is an extreme point of the set of interim probabilities.

Fix a hierarchical interim allocation p and the ranking function R corresponding to the P that generates it. Given any rank $k < K$, let $i(k)$ denote the unique agent i with a type t_i satisfying $R(t_i) = k$ and let $\hat{T}(k)$ denote the set of $t_i \in T_{i(k)}$ with $R(t_i) = k$.

Suppose, contrary to what we wish to show, that p is not an extreme point of the set of interim probabilities. Then there exist interim probabilities q^1 and q^2, neither equal to p, and $\lambda \in (0, 1)$ such that $\lambda q^1 + (1 - \lambda)q^2 = p$. We obtain a contradiction by showing that we must have $q^1 = q^2 = p$.

Clearly, if $K = 1$, there is only one rank and all types of all agents have rank K. In this case, p is the zero vector, so the only interim probabilities q^1 and q^2 that could satisfy $\lambda q^1 + (1 - \lambda)q^2 = p$ for $\lambda \in (0, 1)$ are also the zero vector, establishing our claim.

So assume $K \geq 2$. Fix any $t_{i(1)} \in \hat{T}(1)$. Then $p_{i(1)}(t_{i(1)}) = 1$, so $\lambda q^1 + (1 - \lambda)q^2 = p$ implies $q^j_{i(1)}(t_{i(1)}) = 1$ for $j = 1, 2$.

This initiates an induction. Let K be the number of ranks. Suppose we have shown that for all $k \leq \bar{k} < K$, we have

$$q^1_{i(k)}(t_{i(k)}) = q^2_{i(k)}(t_{i(k)}) = p_{i(k)}(t_{i(k)}),\ \forall t_{i(k)} \in \hat{T}(k).$$

We now show the same is true for rank $k = \bar{k} + 1$. This is obvious if $\bar{k} + 1 = K$ since

$p_i(t_i) = 0$ for any t_i with rank K. So suppose $\bar{k}+1 < K$. Let $i = i(\bar{k}+1)$ and fix any $t_i^* \in \hat{T}(\bar{k}+1)$.

We have

$$p_{i(k)}(t_{i(k)}) = \Pr\left(t_{i(j)} \notin \hat{T}(j),\ j = 1, \ldots, k-1\right)$$

and

$$p_i(t_i^*) = \Pr\left(t_{i(k)} \notin \hat{T}(k),\ k = 1, \ldots, \bar{k}\right).$$

Consider the inequality stated in Lemma 1 for the sets $\hat{T}(k)$, $k = 1, \ldots, \bar{k}$, and $\{t_i^*\}$. (If some agent j has no type in one of these sets, then $\hat{T}_j = \emptyset$.) The left–hand side is

$$\sum_{k=1}^{\bar{k}} \sum_{t_{i(k)} \in \hat{T}(k)} \hat{p}_{i(k)}(t_{i(k)})\mu_{i(k)}(t_{i(k)}) + \hat{p}_i(t_i^*)\mu_i(t_i^*)$$

or

$$\sum_{k=1}^{\bar{k}} \mu_{i(k)}(\hat{T}(k)) \Pr\left(t_{i(j)} \notin \hat{T}(j),\ j = 1, \ldots, k-1\right) + \mu_i(t_i^*) \Pr\left(t_{i(k)} \notin \hat{T}(k),\ k = 1, \ldots, \bar{k}\right).$$

The first term is exactly the probability that one of the agents has a rank of $\bar{k}$ or higher. So the total probability is the probability that either one of the agents has a rank of $\bar{k}$ or higher or else i is type t_i^*.

The right–hand side of the inequality is 1 minus the probability that no type is in one of these sets. That is, the right–hand side is

$$\leq 1 - \Pr(t_{i(k)} \notin \hat{T}(k),\ k \leq \bar{k},\ \text{and } t_i \neq t_i^*).$$

This must hold with equality. The first expression is exactly the probability that one of these types materializes, while the second is 1 minus the probability that none of them do.

Because the inequality holds with equality, we see that given the way we specified q^j on the types ranked above $\bar{k}$, we cannot set $q_i^j(t_i^*) > p_i(t_i^*)$ for either j since doing so would give an interim probability that violates Lemma 1. Hence we again have $q^j(t_i^*) = p_i(t_i^*)$ for $j = 1, 2$, completing the induction.

Hence every hierarchical interim probability is an extreme point of the set of hierarchical probabilities. Next, we show the converse: every extreme point of the set of interim probabilities is a hierarchical interim probability.

To show this, suppose not. Then there must be some interim probability, say p, which is not in the convex hull of the set of hierarchical interim probabilities. Let W denote this convex hull. Since W is convex, there is a separating hyperplane f^*. In other words, viewing p and the elements of W as vectors, there exists a vector f^* such that $f^* \cdot \hat{p} > f^* \cdot q$ for all $q \in W$. Define f to be the vector with nth element $f^*_n/\mu(n)$ where f^*_n is the nth element of f^* and $\mu(n)$ is the probability of the type in the nth position in these vectors.

Without loss of generality, we can assume that the f_n's are all distinct. That is, we have $f_n \neq f_m$ for $n \neq m$. (If not, we can perturb f^* slightly to achieve this property.) Recall that the allocation that never gives the good to any agent is hierarchical. Hence the zero vector is contained in W. Hence $f^* \cdot \hat{p} > 0$ so $f^*_n > 0$ for some n and hence $f_n > 0$ for some n.

Without loss of generality, order the components of vectors so that $f_1 > f_2 > \ldots > f_N$, so we know that $f_1 > 0$. Hence there is some n^* with $f_n > 0$ for $n \leq n^*$ and $f_n \leq 0$ for $n \geq n^* + 1$ where n^* is the length of f if all components are positive.

We construct a hierarchical allocation and the associated $q \in W$ as follows. Define the ranking R as follows. For $n \leq n^*$, assign rank n to the type in the nth component of these vectors. For every $n \geq n^* + 1$, assign rank K to the type in the nth component. Define functions $i(k)$ and $\hat{T}(k)$ for this ranking as above.

The corresponding q has 1 in the first component, $\Pr(t_{i(1)} \notin \hat{T}(1))$ in the second, etc., and has 0 in all components from $n^* + 1$ onward. We now show a contradiction to $f^* \cdot p > f^* \cdot q$.

We can write $f^* \cdot p > f^* \cdot q$ as

$$\sum_{n=1}^{N} f_n \mu(n) p(n) > \sum_{n=1}^{N} f_n \mu(n) q(n) = \sum_{n=1}^{n^*} f_n \mu(n) q(n)$$

where $p(n)$ is the nth component of the vector p and other terms are defined analogously. Equivalently,

$$\sum_{n=1}^{N} f_n \mu(n)(p(n) - q(n)) > 0.$$

Since $f_1 > 0$, this implies

$$\sum_{n=2}^{N} \frac{f_n}{f_1} \mu(n)(p(n) - q(n)) > \mu(1)(q(1) - p(1)).$$

But $q(1) = 1 \geq p(1)$, so this implies

$$\sum_{n=2}^{N} \frac{f_n}{f_1} \mu(n)(p(n) - q(n)) > 0.$$

If $f_2 \leq 0$, this is a contradiction, since we would then have $p(n) \geq 0 = q(n)$ and $f_n \leq 0$ for all $n \geq 2$. So assume $f_2 > 0$.

By assumption, $f_1/f_2 > 1$. Hence

$$\frac{f_1}{f_2} \sum_{n=2}^{N} \frac{f_n}{f_1} \mu(n)(p(n) - q(n)) > \sum_{n=2}^{N} \frac{f_n}{f_1} \mu(n)(p(n) - q(n)) > \mu(1)(q(1) - p(1)).$$

That is,

$$\sum_{n=2}^{N} \frac{f_n}{f_2} \mu(n)(p(n) - q(n)) > \mu(1)(q(1) - p(1)),$$

so

$$\sum_{n=3}^{N} \frac{f_n}{f_2} \mu(n)(p(n) - q(n)) > \mu(2)(q(2) - p(2)) + \mu(1)(q(1) - p(1)).$$

It is not hard to see that the right–hand side must be non–negative. This follows from the fact that the inequality in Lemma 1 implies that $\mu(1)q(1) + \mu(2)q(2)$ equals the maximum possible value for this sum. Hence $\mu(1)p(1) + \mu(2)p(2)$ must be weakly smaller. Hence

$$\sum_{n=3}^{N} \frac{f_n}{f_2} \mu(n)(p(n) - q(n)) > 0.$$

Clearly, iterating, we obtain a contradiction. ▌

Remark 1. Theorem 1 is slightly stronger than what we use. We only need the fact that every extreme point of the set of interim probabilities is a hierarchical interim probability, not the converse. We include the converse for the sake of completeness.