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Hierarchies of Beliefs and Common Knowledge*

ADAM BRANDENBURGER

Harvard Business School, Boston, Massachusetts 02163

AND

EDDIE DEKEL

Department of Economics, University of California at Berkeley, Berkeley, California 94720

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Game-theoretic analysis often leads to consideration of an infinite hierarchy of beliefs for each player. Harsanyi suggested that such a hierarchy of beliefs could be summarized in a single entity, called the player's *type*. This paper provides an elementary construction, complementary to the construction already given in [J-F. Mertens and S. Zamir, Formulation of Bayesian analysis for games with incomplete information, *Int. J. Game Theory* **14** (1985), 1–29] of Harsanyi's notion of a type. It is shown that if a player's type is *coherent* then it induces a belief over the types of the other players. Imposing common knowledge of coherency closes the model of beliefs. We go on to discuss the question that often arises as to the sense in which the structure of a game-theoretic model is, or can be assumed to be, common knowledge. *Journal of Economic Literature* Classification Number: 026.

1. INTRODUCTION

Hierarchies of beliefs arise in an essential way in many problems in decision and game theory. For example, the analysis of a game, even of one with complete information, leads to consideration of an "infinite regress" in beliefs. Thus, supposing for simplicity that there are just two players i and j, the choice of strategy by i will depend on what i believes j's choice

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will be, which in turn will depend on what i believes j believes i's choice will be, and so on. An infinite regress of this kind underlies the idea of "rationalizable" strategies, introduced by Bernheim [5] and Pearce [19]. For games with complete information, this regress in beliefs has traditionally been "cut through" by the imposition of an equilibrium concept such as Nash equilibrium. It was in the context of games with incomplete information, in which some parameters of the game are not common knowledge among the players, that the problem of an infinite regress in beliefs was first tackled by Harsanyi [15]. Harsanyi's solution was to summarize the entire stream of beliefs of a player in a single entity, called the player's *type*, such that each type induces a belief over the types of the other players. Harsanyi's formulation of a game with incomplete information has become an indispensable tool in many areas of economics, but it is only relatively recently that rigorous arguments have been given in support of Harsanyi's notion of a type by Armbruster and Böge [1], Böge and Eisele [7], and Mertens and Zamir [18]. This paper provides an alternative construction of types, which is similar to that in [18] but which relies on more elementary mathematics and is more explicit about what is assumed to be common knowledge.

Our construction of types has two stages. First, we show that if an individual's type is *coherent* then it induces a belief over the types of the other individuals. (Coherency is a requirement that the various levels of beliefs of an individual do not contradict one another—see Definition 1.) This result (Proposition 1) is essentially just a statement of Kolmogorov's Existence Theorem from the theory of stochastic processes (see, for example, Chung [9, p. 60]). Second, the model of beliefs is closed by imposing, via a simple inductive definition, the requirement that each type knows (in the probabilistic sense of assigning probability 1) that the other individuals' types are coherent, that each type knows that the other types know this, and so on. That is, the model is closed by imposing common knowledge of coherency. (What is meant here by "closed" is elucidated in the paragraph preceding Definition 1.)

At a technical level, we replace the assumption in Mertens and Zamir that the underlying state space is compact with the assumption that it is complete separable metric. (Remark 2.18 in Mertens and Zamir suggests that such a replacement is possible.) Recently, Heifetz [16] has provided a general construction of types, assuming only that the underlying state space is Hausdorff.

Having completed our construction of types in Section 2, we go on in Section 3 to discuss the question that often arises as to the sense in which the structure of a game-theoretic model is, or can be assumed to be, common knowledge. This question has been discussed by Aumann and others; see [2, 3, 4, 12, 17, 20, 21, 22]. Aumann has argued that if the structure is not common knowledge, then the description of the states of the world is *incomplete* and so the state space should be expanded. The construction of types shows what the expanded state space should be, namely the product of the underlying state space and the individuals' type spaces. Moreover, on the expanded state space, common knowledge of the structure is captured by the assumption of common knowledge of coherency.

2. Construction of Types

In this section hierarchies of beliefs are constructed. The notions of type and coherency are defined and it is shown that a coherent type induces a belief over other individuals' types. We go on to prove that common knowledge of coherency closes the model of beliefs in the sense that all beliefs are then completely specified.

There are two individuals i and j who face some common (underlying) space of uncertainty S.[1] The space S is assumed to be complete separable metric (Polish). For any metric space Z let $\Delta(Z)$ denote the space of probability measures on the Borel field of Z, endowed with the weak topology. According to Bayesian decision theory, each individual must have a belief over the space S; the individuals' first-order beliefs are then elements of $\Delta(S)$. Since each individual may not know the belief of the other, each must have a second-order belief. That is, i's second-order belief is a joint belief over S and the space of j's first-order beliefs; i's second-order belief is thus an element of $\Delta(S \times \Delta(S))$. Similarly for j. Formally, define spaces

$$X_0 = S$$

$$X_1 = X_0 \times \Delta(X_0)$$

$$\vdots$$

$$X_n = X_{n-1} \times \Delta(X_{n-1})$$

$$\vdots$$

A *type* t^i of i is just a hierarchy of beliefs $t^i = (\delta_1^i, \delta_2^i, \ldots) \in \times_{n=0}^{\infty} \Delta(X_n)$. Similarly for j. Let $T_0 = \times_{n=0}^{\infty} \Delta(X_n)$ denote the space of all possible types of i or j.

Of course, i only knows his own type and not the type of j. (Likewise for j.) So it seems that a "second level" hierarchy of beliefs is required, wherein i has a belief over j's type, over j's belief over i's type, and so on. Thus, in the absence of further assumptions, a model which specifies only the hierarchy of beliefs $(\delta_1^i, \delta_2^i, \ldots) \in \times_{n=0}^{\infty} \Delta(X_n)$ for i, and likewise for j, is

[1] All our arguments generalize immediately to the case of more than two individuals.

not closed. The condition under which the specification of i's type already determines his belief over j's type is defined next.

DEFINITION 1. A type $t = (\delta_1, \delta_2, \ldots) \in T_0$ is *coherent* if for every $n \geq 2$, $\text{marg}_{X_{n-2}} \delta_n = \delta_{n-1}$, where $\text{marg}_{X_{n-2}}$ denotes the marginal on the space X_{n-2}.

Coherency says that the different levels of beliefs of an individual do not contradict one another.[2] Let T_1 denote the set of all coherent types. The following proposition shows that a coherent type induces a belief over S and the space of types of the other individual.

PROPOSITION 1. *There is a homeomorphism* $f\colon T_1 \to \Delta(S \times T_0)$.

Proposition 1 will be an easy consequence of the following lemma, which itself is essentially a statement of Kolmogorov's Existence Theorem.

LEMMA 1. *Suppose* $\{Z_n\}_{n=0}^{\infty}$ *is a collection of Polish spaces, and let*

$$D = \{(\delta_1, \delta_2, \ldots)\colon \delta_n \in \Delta(Z_0 \times \cdots \times Z_{n-1})\ \forall n \geq 1,$$
$$\text{marg}_{Z_0 \times \cdots \times Z_{n-2}} \delta_n = \delta_{n-1}\ \forall n \geq 2\}.$$

Then there is a homeomorphism $f\colon D \to \Delta(\times_{n=0}^{\infty} Z_n)$.

Proof. Consider any element $(\delta_1, \delta_2, \ldots) \in D$. By a version of Kolmogorov's Existence Theorem [10, p. 68] there is a unique measure $\delta \in \Delta(\times_{n=0}^{\infty} Z_n)$ such that $\text{marg}_{Z_0 \times \cdots \times Z_{n-1}} \delta = \delta_n$ for all $n \geq 1$. Let f map $(\delta_1, \delta_2, \ldots)$ into this δ. The map f is 1–1 since the value of δ on the cylinders is given by the δ_n's; f is onto since given any $\delta \in \Delta(\times_{n=0}^{\infty} Z_n)$, $f(\text{marg}_{Z_0} \delta, \text{marg}_{Z_0 \times Z_1} \delta, \ldots) = \delta$. Note that $f^{-1}(\delta) = (\text{marg}_{Z_0} \delta, \text{marg}_{Z_0 \times Z_1} \delta, \ldots)$ so f^{-1} is continuous since the maps $\delta \mapsto \text{marg}_{Z_0 \times \cdots \times Z_{n-1}} \delta$, $n \geq 1$, are all continuous. To see that f is continuous, consider a sequence $(\delta_1^r, \delta_2^r, \ldots) \to (\delta_1, \delta_2, \ldots)$ in D, i.e., δ_n^r converges weakly to δ_n for all $n \geq 1$. Let $\delta^r = f(\delta_1^r, \delta_2^r, \ldots)$, $\delta = f(\delta_1, \delta_2, \ldots)$. We have to show that δ^r converges weakly to δ. But this follows from the fact that the cylinders form a convergence-determining class and the values of δ^r, δ on the cylinders are given by the δ_n^r's, δ_n's, respectively. ∎

Proof of Proposition 1. In Lemma 1, set $Z_0 = X_0$, $Z_n = \Delta(X_{n-1})$ for $n \geq 1$. So $Z_0 \times \cdots \times Z_n = X_n$ and $\times_{n=0}^{\infty} Z_n = S \times T_0$. If S is a Polish space then so is $\Delta(S)$ (Dellacherie and Meyer [10, p. 73]), hence the Z_n's will be

[2] What is here called coherency is usually called consistency in the theory of stochastic processes. The term coherency is used to avoid confusion with Harsanyi's use of the term consistency, which means something different.

Polish spaces provided S is. The set of coherent types T_1 is exactly D. So Lemma 1 implies that there is a homeomorphism f: $T_1 \to \Delta(S \times T_0)$. ∎

An obvious question to ask is why the particular homeomorphism f, just constructed, is "natural." The reason is the following property of f: the marginal probability assigned by $f(\delta_1, \delta_2, ...)$ to a given event in X_{n-1} is equal to the probability that δ_n assigns to that same event. That is, in deriving probabilities on the product space $S \times T_0 = X_0 \times \Delta(X_0) \times \Delta(X_1) \times \cdots$ from $(\delta_1, \delta_2, ...)$, the function f preserves the probabilities specified by each δ_n on each X_{n-1}.

Coherency implies that i's type determines i's belief over j's type. But i's type does not necessarily determine i's belief over j's belief over i's type—in particular this is so if i believes it possible that j's type is not coherent. For a type to determine *all* beliefs (including beliefs over beliefs over types), common knowledge of coherency must be imposed. To do so, define a sequence of sets T_k, $k \geq 2$, by

$$T_k = \{t \in T_1 : f(t)(S \times T_{k-1}) = 1\}.$$

(It is straightforward to show inductively that T_{k-1} is a Borel set, so T_k is indeed well defined.) Let $T = \bigcap_{k=1}^{\infty} T_k$. The set $T \times T$ is the subset of $T_1 \times T_1$ obtained by requiring the following statements to hold: (1) i knows j's type is coherent; (2) j knows i's type is coherent; (3) i knows j knows i's type is coherent; and so on. That is, $T \times T$ is the set of types which satisfy common knowledge of coherency. The following proposition shows that the space T closes the model, and corresponds to the "universal type space" of Theorem 2.9 in [18].

PROPOSITION 2. *There is a homeomorphism* g: $T \to \Delta(S \times T)$.

Proof. It is easy to check that $T = \{t \in T_1 : f(t)(S \times T) = 1\}$, so $f(T) = \{\delta \in \Delta(S \times T_0): \delta(S \times T) = 1\}$ since f is onto. But $f(T)$ is homeomorphic to T and $\{\delta \in \Delta(S \times T_0): \delta(S \times T) = 1\}$ is homeomorphic to $\Delta(S \times T)$ (for any metric space Z and measurable subset W of Z, $\{\delta \in \Delta(Z): \delta(W) = 1\}$ is homeomorphic to $\Delta(W)$). So T is homeomorphic to $\Delta(S \times T)$. ∎

Once again an immediate question arises as to why the homeomorphism g is "natural." The answer is that g preserves the beliefs of each individual in exactly the same way as the function f of Proposition 1 preserves beliefs. (See the discussion following the proof of Proposition 1.) Moreover, the development in Section 3—where we show how the model of hierarchies of beliefs can be transformed into a standard model of differential information—uses the homeomorphism g, and, in particular, Proposition 3 relies essentially on the specific homeomorphism g.

A technical aspect of the construction worth noting is that closure of the model of hierarchies of beliefs is not a purely measure-theoretic result. Recall in fact that we assumed S to be a Polish space. This is because (cf. [13, pp. 211–212]) Kolmogorov's Existence Theorem is itself not purely measure-theoretic, and relies on topological assumptions.

3. Relationship to the Standard Model of Differential Information

The standard formulation of a model of differential information as commonly used in game theory and economics is a collection $\langle \Omega, H^i, H^j, p^i, p^j \rangle$.[3] The set Ω is the space of states of the world, H^i is i's information partition (if $\omega \in \Omega$ is the true state, i is informed of the cell of H^i that contains ω), p^i is i's prior probability measure on Ω, and H^j and p^j are the analogous objects for j. In this section we discuss the relationship between the standard formulation and the types model constructed in Section 2. First, we use the types model to shed some light on the interpretational question mentioned in the Introduction, namely the sense in which the structure of the standard model is, or can be assumed to be, common knowledge. Second, we describe a transformation of the types model into a stantard model and demonstrate, by way of example, that the transformation is meaningful.

An interpretational question that often arises in discussions of the standard model of differential information is whether the information structure (consisting of partitions and priors) is "common knowledge" is an informal sense. (We say in an informal sense because the information structure is not an event in Ω and hence the formal definition of common knowledge does not apply. In what follows we will use quotation marks when we wish to signify informal usage.) The issue of "common knowledge" of the information structure arises in the following manner. Given an event A in Ω one can define, using i's information structure H^i and p^i, the event that i knows A (see, e.g., [2]), to be denoted $K^i(A)$. Similarly for j. Now suppose in fact that $A = K^j(B)$ for some event B in Ω. Then $K^i(A) = K^i(K^j(B))$ is certainly interpretable as the event that i knows $K^j(B)$. But in practice we interpret $K^i(K^j(B))$ as the event that i knows j knows B—and this latter interpretation relies on an implicit assumption that i "knows" j's information structure. That is, it is assumed that i "knows" H^j and p^j. Applying the same argument to more complex events such as $K^i(K^j(K^i(C)))$ and the like shows that in fact "common knowledge" of the information structure is needed.

The nature of this "common knowledge" has been much discussed by

[3] We maintain the simplifying assumption of only two individuals i and j.

Aumann and others; see [2, 3, 4, 12, 17, 20, 21, 22].[4] Aumann has argued that "common knowledge" of the information structure is without loss of generality since the description of a state in Ω should include a description of the manner in which information is imparted to the individuals (the partitions) and a description of the players' beliefs (the priors). If this is not the case, Aumann argues that the description of the states is incomplete and so the state space should be expanded. The observation we wish to make is that the appropriate expanded state space is the product of the underlying state space S and the type spaces T. More precisely, the expanded state space is $S \times T \times T$ where the first copy of T is the type space of individual i and second copy of T is the type space of individual j.[5] The point is that "common knowledge" of the information structure on the expanded state space is captured by the assumption of common knowledge of coherency that we made in Section 2. To see this, consider, for example, the set $T_2 = \{t \in T_1 : f(t)(S \times T_1) = 1\}$ as defined there. The set T_2 is the set of types of i, say, which know that j's type is coherent. So T_2 is the set of types of i which can calculate beliefs over j's beliefs over i's type, or in other words the set of types of i which "know"j's information structure. Similarly, T_3 is the set of types of i which can calculate beliefs over j's beliefs over i's beliefs over j's type, or in other words, the set of types of i which "know" that j "knows" i's information structure. And so on. The upshot is that since common knowledge of coherency is a natural rationality assumption (it merely states that it is common knowledge that the various levels of beliefs of an individual do not contradict one another), "common knowledge" of the information structure (on the expanded state space) is indeed without loss of generality.

To summarize, the same model, namely the model of hierarchies of beliefs, validates Harsanyi's notion of a type and Aumann's notion of a space of completely specified states of the world.

We now show how to transform the types model into a standard model, thus demonstrating that the standard model is in fact no less general than the types model. From this, it follows that the standard model, which is, of course, a simpler construct, can be employed whenever doing so is more convenient.

Starting with an underlying space of uncertainty S and induced type spaces T, we can construct a standard model as follows. The set Ω of states of the world is the product space $S \times T \times T$, where, as before, the first copy of T is the type space of individual i and the second copy of T is the type space of individual j. Note that even if S is finite, $S \times T \times T$ is an

[4] There are also relevant literatures in computer science, artificial intelligence, linguistics, and philosophy; see [11, 14, 23], and the references therein.

[5] Thus $\Omega = S \times T \times T$. A formal treatment of the information structure on Ω is given below.

uncountable space and the information structure on $S \times T \times T$ must be specified in terms of σ-fields rather than partitions. Let $\mathscr{H}$ denote the Borel field of $S \times T \times T$. Since the information that i possesses is exactly knowledge of his own type, the natural sub σ-field of $\mathscr{H}$ for i is $\{S \times B \times T : B \text{ is a Borel subset of } T\}$. The homeomorphism g of Proposition 2 determines i's beliefs: the natural conditional probability for i to access to an event $A \in \mathscr{H}$ at a state (s, t^i, t^j) is $g(t^i)(A_{t^i})$ where $A_{t^i} = \{(s, t^j) : (s, t^i, t^j) \in A\}$. Individual j's sub σ-field and beliefs are specified in analogous fashion. In sum, we have shown, with one proviso about to be discussed, how the types model can be transformed into a standard model. The proviso is that we have specified i and j's system of *conditional* probability measures rather than their *prior* probability measures. In fact, there would be no difficulty in constructing for i and j (different) prior probabitity measures on $S \times T \times T$ with the indicated conditionals. (The technical conditions allowing this are readily verified.) But since it is the conditionals, and not the priors, that are of decision-theoretic significance, we refrain from going into details on constructing the priors.[6]

So far, we have shown how the types model can, formally speaking, be transformed into a standard model. That this is a sensible way of viewing the types model is best seen by working through an example. Suppose we wish to write down the statement that an event is *common knowledge* between i and j. There is a natural way, which we give in a moment, of doing this in the context of the types model. There is also the well known definition, due to Aumann [2], of common knowledge in the context of a standard model of differential information. What we are going to do is to show that the "types" definition of common knowledge is equivalent to the "standard" definition of common knowledge when the latter is applied to the standard model derived from the types model in the manner described in the preceding paragraph.

We start with the "types" definition. Given an event E in S, let

$$V_1(E) = \{t \in T : g(t)(E \times T) = 1\}$$

and then define a sequence of sets $V_k(E)$, $k \geqslant 2$, by

$$V_k(E) = \{t \in T : g(t)(S \times V_{k-1}(E)) = 1\}.$$

(It is straightforward to show inductively that $V_{k-1}(E)$ is a Borel set, so $V_k(E)$ is indeed well defined.) Let $V(E) = \bigcap_{k=1}^{\infty} V_k(E)$. Then we say that E is common knowledge between i and j according to the "types" definition if $(t^i, t^j) \in V(E) \times V(E)$. This definition simply states that i is of a type that assigns probability 1 to E, j is of a type that assigns probability 1 to E,

[6] Also worth noting is that the constructed conditionals are regular and proper (the latter in the sense of Blackwell and Dubins [6]).

i is of a type that assigns probability 1 to j being of a type that assigns probability 1 to E, and so on.

We now turn to the "standard" definition of common knowledge. Aumann's original definition was couched in terms of partitions. However, as was pointed out above, the set $\Omega = S \times T \times T$ is uncountable and σ-fields rather than partitions must be employed. A generalization of Aumann's definition to cover this case was proposed in [8] and we follow this approach here. The event that i knows an event $A \in \mathscr{H}$, to be denoted by $K^i(A)$, is given by

$$K^i(A) = \{(s, t^i, t^j): g(t^i)(A_{t^i}) = 1\}.$$

The event that j knows A, to be denoted by $K^j(A)$, is defined in analogous fashion. Thus $K(A) = K^i(A) \cap K^j(A)$ is the event that everyone knows A. We say that A is common knowledge at a state (s, t^i, t^j) according to the "standard" definition if $(s, t^i, t^j) \in K_\infty(A)$, where K_∞ denotes the infinite application of the K operator.

It remains to establish that the event E in S is common knowledge according to the "types" definition if and only if the event $E \times T \times T$ in Ω is common knowledge according to the "standard" definition.[7] The equivalence is stated formally in the following proposition.

PROPOSITION 3. $S \times V(E) \times V(E) = K_\infty(E \times T \times T)$.

Proof. The proof follows immediately from the definitions. Observe that

$$K^i(E \times T \times T) = \{(s, t^i, t^j): g(t^i)(E \times T) = 1\} = S \times V_1(E) \times T.$$

Similarly, $K^j(E \times T \times T) = S \times T \times V_1(E)$ and hence $K(E \times T \times T) = S \times V_1(E) \times V_1(E)$. Continuing in this fashion establishes that $K_\infty(E \times T \times T) = S \times V(E) \times V(E)$. ∎

Proposition 3 confirms that our transformation of the types model into a standard model makes sense although, strictly speaking, this has been shown to be true only insofar as common knowledge of events is concerned. Nevertheless, it should be clear that in fact *any* calculation involving the individuals' beliefs is preserved under the transformation.

The reserve transformation has been considered by Tan and Werlang [22]. They show how, starting from the standard formulation of a model of differential information, to calculate the induced hierarchies of beliefs and hence how to construct an associated types model. They also

[7] Note that the equivalence relates E in S to $E \times T \times T$ in Ω. This is because the set E is not an event in Ω, but is naturally identified with the event $E \times T \times T$.

demonstrate that their transformation is meaningful by showing that it preserves the notion of common knowledge.

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