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LEXICOGRAPHIC PROBABILITIES AND CHOICE UNDER UNCERTAINTY

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Two properties of preferences and representations for choice under uncertainty which play an important role in decision theory are: (i) admissibility, the requirement that weakly dominated actions should not be chosen; and (ii) the existence of well defined conditional probabilities, that is, given any event a conditional probability which is concentrated on that event and which corresponds to the individual's preferences. The conventional Bayesian theory of choice under uncertainty, subjective expected utility (SEU) theory, fails to satisfy these properties—weakly dominated acts may be chosen, and the usual definition of conditional probabilities applies only to non-null events. This paper develops a non-Archimedean variant of SEU where decision makers have lexicographic beliefs; that is, there are (first-order) likely events as well as (higher-order) events which are infinitely less likely but not necessarily impossible. This generalization of preferences, from those having an SEU representation to those having a representation with lexicographic beliefs, can be made to satisfy admissibility and yield well defined conditional probabilities and at the same time to allow for "null" events. The need for a synthesis of expected utility with admissibility, and to provide a ranking of null events, has often been stressed in the decision theory literature. Furthermore, lexicographic beliefs are appropriate for characterizing refinements of Nash equilibrium. In this paper we discuss: axioms on, and behavioral properties of, individual preferences which characterize lexicographic beliefs; probability-theoretic properties of the representations; and the relationships with other recent extensions of Bayesian SEU theory.

KEYWORDS: Admissibility, weak dominance, conditional probabilities, lexicographic probabilities, non-Archimedean preferences, subjective expected utility.

1. INTRODUCTION

THERE ARE TWO IMPORTANT properties of preferences and representations for choice under uncertainty. The first is the criterion of admissibility, namely, that a decision maker should not select a weakly dominated action (Luce and Raiffa (1957, Chapter 13)). The second property is that for *any* event there is a conditional probability that is concentrated on that event and that represents the decision maker's conditional preferences given that event. We call such conditional probabilities "well defined." The importance for a complete and intuitive theory to provide conditional probabilities given any event has long been discussed in the context of probability theory[2] and philosophy.[3] Moreover, the criterion of backwards induction, which specifies that at every choice node

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[2] For example de Finetti (1972, p. 82) says "there seems to be no justification ... for introducing the restriction $P(H) \neq 0$", where H is a conditioning event, and Blackwell and Dubins (1975, p. 741) are concerned with when "conditional distributions ... satisfy the intuitive desideratum ... of being *proper*," that is, concentrated on the conditioning event.

[3] See Harper, Stalnacker, and Pearce (1981) as well as the references cited there for a discussion of the relationship between linguistic intuition, counterfactuals, and conditional probabilities.

in a decision tree choices maximize expected utility with respect to "beliefs" at that node, requires the use of well defined conditional probabilities in representing conditional preferences at every node in the tree. Nonetheless, conventional subjective expected utility (SEU) theory does not satisfy these properties.[4] For both properties, the source of the problem is the same: conditional on events which are not expected to occur, SEU theory leads to trivial choice problems—all acts are indifferent.

Obviously conventional SEU theory can be refined to satisfy admissibility and to determine well defined conditional probabilities by ruling out null events. But this method is too restrictive. For example, such preferences could not characterize pure strategy equilibria in games. In this paper we axiomatize preferences under uncertainty which both satisfy admissibility and yield well defined conditional probabilities, yet allow for events which are "null," although not in the sense of Savage (1954). We develop a non-Archimedean SEU theory starting from Fishburn's (1982) version of the SEU framework due to Anscombe and Aumann (1963). A key feature of our representation is the introduction of a lexicographic hierarchy of beliefs. Such beliefs can capture the idea that a die landing on its side is infinitely more likely than its landing on an edge, which in turn is infinitely more likely than its landing on a corner. These considerations of "unlikely" events might seem rather arcane, but are nevertheless crucial in a game-theoretic context, as the discussion above suggests and as is shown in the sequel to this paper (Blume, Brandenburger, and Dekel (1990)). In particular, a player in a game may be unwilling to exclude *entirely* from consideration any action of an opponent, and, moreover, which actions are unlikely is in a sense "endogenous" (i.e. depends on the equilibrium under consideration).

These objectives motivate our first and main departure from the axioms of SEU—weakening the Archimedean axiom. After a review of SEU in Section 2, we present a general representation theorem for preferences with the weakened Archimedean axiom in Section 3. This representation allows for events which are "null," yet are taken into consideration by the decision maker, as well as events which are null in the sense of Savage. We then strengthen the state independence axiom to rule out Savage-null events, and in Section 4 we show that these preferences satisfy admissibility and determine well defined conditional probabilities. Section 5 discusses an Archimedean axiom intermediate in strength between the standard Archimedean axiom and the Archimedean axiom of Section 3. This intermediate axiom leads to a representation of choice which is closely related to conditional probability systems (Myerson (1986a, b)). In Section 6 an alternative representation of preferences, which is equivalent to that of Section 3, is provided using infinitesimal numbers instead of a lexicographic order of vectors. Section 7 discusses the surprisingly delicate issue of modelling stochastic independence with lexicographic probabilities.

[4] Some early work in statistical decision theory was concerned with the problematic relationship between Bayes procedures and admissible procedures. See Blackwell and Girshick (1954, Section 5.2) and Arrow, Barankin, and Blackwell (1953).

It is worth emphasizing that the notion of lexicographic beliefs arises in a very natural fashion—as a consequence of satisfying the decision-theoretic properties of admissibility and the existence of well defined conditional probabilities on all events, together with allowing for some kind of null events. Lexicographic models have been used to explore other issues in decision theory. An early example is Chipman (1960, 1971a, b), who developed and applied them to provide an alternative to the Friedman and Savage (1948) explanation of gambling and insurance purchases, and to discuss portfolio choice and other economic applications. Fishburn (1974) provides a comprehensive survey. Kreps and Wilson (1982) introduced a lexicographic method of updating beliefs in game trees in the context of sequential equilibrium.[5] Hausner (1954) and Richter (1971) provide the technical foundations for the work we present here.

2. SUBJECTIVE EXPECTED UTILITY ON FINITE STATE SPACES

There are two distinct approaches to the theory of subjective expected utility. In Savage's (1954) framework individuals have preferences over acts which map a state space into consequences. Anscombe and Aumann (1963) (as well as Chernoff (1954), Suppes (1956), and Pratt, Raiffa, and Schlaifer (1964)) use axioms which refer to objective probabilities. Although the Savage (1954) framework is perhaps more appealing, for reasons of tractability and because we will want to apply our results to finite games, we employ the Anscombe and Aumann framework in Section 3 to develop our non-Archimedean SEU theory. To facilitate subsequent comparisons, a brief review of Anscombe and Aumann's SEU theory follows.

The decision maker faces a finite set of states Ω and a set of (pure) consequences C. Let $\mathscr{P}$ denote the set of simple (i.e. finite support) probability distributions on consequences. The objective lotteries in $\mathscr{P}$ provide a scale for measuring the utilities of consequences and the subjective probabilities of states. The decision maker has preferences over acts, which are maps from the state space Ω into $\mathscr{P}$. Thus the set of acts is the product space $\mathscr{P}^{\Omega}$. The ωth coordinate of act x is denoted x_ω. The interpretation of an act x is that when it is chosen, the consequence for the decision maker if state ω occurs is determined by the lottery x_ω. The set $\mathscr{P}^{\Omega}$ is a mixture space; in particular, for $0 \leqslant \alpha \leqslant 1$ and $x, y \in \mathscr{P}^{\Omega}$, $\alpha x + (1-\alpha)y$ is the act that in state ω assigns probability $\alpha x_\omega(c) + (1-\alpha)y_\omega(c)$ to each $c \in C$. Nonempty subsets of Ω are termed events. For any event $S \subset \Omega, x_S$ denotes the tuple $(x_\omega)_{\omega \in S}$. We will denote $x_{\Omega - S}$ by x_{-S}. A constant act maps each state into the same lottery on consequences: $x_\omega = x_{\omega'}$ for all $\omega, \omega' \in \Omega$. For notational simplicity we often write ω to represent the event $\{\omega\} \subset \Omega$.

The decision maker's weak preference relation over pairs of acts is denoted by $\succcurlyeq$. The relations of strict preference, denoted $\succ$, and indifference, denoted

[5] The relationship between this paper and the decision-theoretic underpinnings of sequential equilibrium can be seen in our discussion in Section 5 of conditional probability systems. McLennan (1989b) uses conditional probability systems to characterize sequential equilibrium.

$\sim$, are defined by: $x \succ y$ if $x \succcurlyeq y$ and not $y \succcurlyeq x$; and $x \sim y$ if $x \succcurlyeq y$ and $y \succcurlyeq x$. The following axioms characterize those preference orders with (Archimedean) SEU representations.

AXIOM 1 (Order): *$\succcurlyeq$ is a complete and transitive binary relation on $\mathscr{P}^{\Omega}$.*

AXIOM 2 (Objective Independence): *For all $x, y, z \in \mathscr{P}^{\Omega}$ and $0 < \alpha \leqslant 1$, if $x \succ$ (respectively $\sim$)y, then $\alpha x + (1-\alpha)z \succ$ (respectively $\sim$)$\alpha y + (1-\alpha)z$.*

AXIOM 3 (Nontriviality): *There are $x, y \in \mathscr{P}^{\Omega}$ such that $x \succ y$.*

AXIOM 4 (Archimedean Property): *If $x \succ y \succ z$, then there exists $0 < \alpha < \beta < 1$ such that $\beta x + (1-\beta)z \succ y \succ \alpha x + (1-\alpha)z$.*

A definition of null events requires the notion of conditional preferences $\succcurlyeq_S$ for each $S \subset \Omega$, as in Savage (1954).

DEFINITION 2.1: $x \succcurlyeq_S y$ if, for some $z \in \mathscr{P}^{\Omega}$, $(x_S, z_{-S}) \succcurlyeq (y_S, z_{-S})$.

By Axioms 1 and 2 this definition is independent of the choice of z. (This can be seen by assuming to the contrary that: (i) $(x_S, z_{-S}) \succcurlyeq (y_S, z_{-S})$; while (ii) $(y_S, w_{-S}) \succ (x_S, w_{-S})$. Then taking $\frac{1}{2}:\frac{1}{2}$ mixtures of (x_S, w_{-S}) with (i) and of (x_S, z_{-S}) with (ii) and applying Axioms 1 and 2 yields a contradiction.) An event S is Savage-null if its conditional preference relation is "trivial."

DEFINITION 2.2: The event $S \subset \Omega$ is *Savage-null* if $x \sim_S y$ for all $x, y \in \mathscr{P}^{\Omega}$.

AXIOM 5 (Non-null State Independence): *For all states $\omega, \omega' \in \Omega$ which are not Savage-null and for any two constant acts $x, y \in \mathscr{P}^{\Omega}$, $x \succcurlyeq_{\omega} y$ if and only if $x \succcurlyeq_{\omega'} y$.*

The following representation theorem can be found in Anscombe and Aumann (1963) and Fishburn (1982, p. 111, Theorem 9.2).

THEOREM 2.1: *Axioms 1–5 hold if and only if there is an affine function $u: \mathscr{P} \to \mathbb{R}$ and a probability measure p on Ω such that, for all $x, y \in \mathscr{P}^{\Omega}$,*

$$x \succcurlyeq y \Leftrightarrow \sum_{\omega \in \Omega} p(\omega) u(x_{\omega}) \geqslant \sum_{\omega \in \Omega} p(\omega) u(y_{\omega}).$$

Furthermore, u is unique up to positive affine transformations, p is unique, and $p(S) = 0$ if and only if the event S is Savage-null.

Since u is an affine function, and x_{ω} has finite support, $u(x_{\omega}) = \sum_{c \in C} u(\delta_c) x_{\omega}(c)$ where δ_c denotes the measure assigning probability one to c. In order to focus on the subjective probabilities, which are our main concern, and for clarity of the equations, we write u as an affine function on $\mathscr{P}$ as above, rather than including this latter summation explicitly.

COROLLARY 2.1: *If the event S is not Savage-null, then for all $x, y \in \mathscr{P}^{\Omega}$,*

$$x \succcurlyeq_S y \Leftrightarrow \sum_{\omega \in S} p(\omega|S)u(x_\omega) \geqslant \sum_{\omega \in S} p(\omega|S)u(y_\omega).$$

In this corollary, which is immediate from Definition 2.1 and Theorem 2.1, $p(\omega|S)$ is given by the usual definition of conditional probability: $p(\omega|S) = p(\omega \cap S)/p(S)$. Corollary 2.1 applies only to events which are not Savage-null since conditional preferences on Savage-null events are trivial and conditional expected utility given any Savage-null event is not defined. To guarantee admissibility and well defined conditional probabilities, Savage-null events must be ruled out. This can be done by strengthening the non-null state independence axiom.

AXIOM 5′ (State Independence): *For all states $\omega, \omega' \in \Omega$ and for any two constant acts $x, y \in \mathscr{P}^{\Omega}$, $x \succcurlyeq_\omega y$ if and only if $x \succcurlyeq_{\omega'} y$.*

Under Axioms 1–4 and 5′ the same representation as in Theorem 2.1 obtains, with the additional feature that $p(\omega) > 0$ for all $\omega \in \Omega$. The consequence is that all odds ratios are finite. The decision maker *must* trade off utility gains in any one state against utility gains in *any* other state. Our formulation of non-Archimedean SEU theory avoids this. We will have states which are not Savage-null, and yet which are infinitely less likely than other states.

3. LEXICOGRAPHIC PROBABILITY SYSTEMS AND NON-ARCHIMEDEAN SEU THEORY

In this section we undertake the promised weakening of the Archimedean property (Axiom 4). The consequence of weakening this axiom is the introduction of a new class of null events distinct from the class of Savage-null events. The weakened Archimedean axiom does not eliminate the Savage-null events; that is the consequence of strengthening state independence. Thus the decision theory we introduce in this section, non-Archimedean SEU theory, is strictly weaker than the conventional Archimedean SEU theory in that it can rationalize a strictly larger set of choices. Our new axiom is a restriction of the Archimedean property to those triples of acts x, y, z such that $x_{-\omega} = y_{-\omega} = z_{-\omega}$ for some state $\omega \in \Omega$.

AXIOM 4′ (Conditional Archimedean Property): *For each $\omega \in \Omega$, if $x \succ_\omega y \succ_\omega z$, then there exists $0 < \alpha < \beta < 1$ such that $\beta x + (1-\beta)z \succ_\omega y \succ_\omega \alpha x + (1-\alpha)z$.*

As a consequence of this weakening of Axiom 4, a numerical representation of preferences is not always possible. (However, in Section 6 we show that a numerical representation *is* possible if one is willing to interpret "numerical" as including infinitesimals.) Here we assign to each act a vector of expected utilities in a Euclidean space, and order these vectors using the lexicographic ordering, which we denote $\geqslant_L$.[6] The expected utility vectors are calculated by taking expectations of a single utility function with respect to a lexicographic hierarchy of probability distributions.

DEFINITION 3.1: A *lexicographic probability system* (LPS) is a K-tuple $\rho = (p_1, \dots, p_K)$, for some integer K, of probability distributions on Ω.

THEOREM 3.1: *Axioms* 1–3, 4′, *and* 5 *hold if and only if there is an affine function* u: $\mathscr{P} \to \mathbb{R}$ *and an LPS* $(p_1, \dots, p_K)$ *on* Ω *such that, for all* $x, y \in \mathscr{P}^{\Omega}$,

$$x \succsim y \Leftrightarrow \left(\sum_{\omega \in \Omega} p_k(\omega) u(x_\omega) \right)_{k=1}^{K} \geqslant_L \left(\sum_{\omega \in \Omega} p_k(\omega) u(y_\omega) \right)_{k=1}^{K}.$$

Furthermore, u *is unique up to positive affine transformations. There is a minimal* K *less than or equal to the cardinality of* Ω. *Among LPS's of minimal length* K, *each* p_k *is unique up to linear combinations of* $p_1, \dots, p_k$ *which assign positive weight to* p_k. *Finally,* $p_k(S) = 0$ *for all* k *if and only if the event* S *is Savage-null.*

The proof of Theorem 3.1, together with proofs of all subsequent results in the paper, can be found in the Appendix. The restriction in the uniqueness part of the theorem to LPS's of minimal length is made in order to avoid redundancies such as the duplication of levels in the hierarchy. (For example, the LPS's $(p_1, p_2, \dots, p_K)$ and $(p_1, p_1, p_2, \dots, p_K)$ obviously represent the same preferences.) Among LPS's of minimal length K, an LPS $(q_1, \dots, q_K)$ will generate the same preferences as $(p_1, \dots, p_K)$ if and only if each $q_k = \sum_{i=1}^{k} \alpha_i p_i$ where the α_i's are numbers such that $\sum_{i=1}^{k} \alpha_i p_i$ is a probability distribution on Ω and $\alpha_k > 0$. In particular, p_1 is unique.

These preferences include $K = 1$, Archimedean theory, as a special case. The following is an example of non-Archimedean behavior allowed by Axiom 4′ but not by Axiom 4. The decision maker will bet on the throw of a die. She has two levels of beliefs, represented by the probability distributions p_1 and p_2. The state space Ω contains the 6 faces of the die, the 12 edges, and the 8 corners. Let

$$p_1(\omega) = \begin{cases} 1/6 & \text{if } \omega \text{ is a face,} \\ 0 & \text{otherwise,} \end{cases} \qquad p_2(\omega) = \begin{cases} 1/12 & \text{if } \omega \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Consider now two bets. Bet x pays off \$$v$ if the die lands on the face labelled 1, and nothing otherwise. Bet y pays off \$1 if the die lands on the face labelled 2 or on any edge, and 0 otherwise. The decision maker's utility function is $u(w) = w$. With these preferences and lexicographic beliefs, $x \succ y$ whenever $v > 1$, and $y \succ x$ whenever $v \leqslant 1$. Notice that there is no v such that $x \sim y$. This

[6] For $a, b \in \mathbb{R}^K$, $a \geqslant_L b$ if and only if whenever $b_k > a_k$, there exists a $j < k$ such that $a_j > b_j$.

type of behavior cannot be rationalized by Archimedean SEU theory since it explicitly violates Axiom 4. Each face occurs with positive first order probability, and each edge occurs with positive second order probability. However, the die landing on a corner is a Savage-null event, hence is assigned probability 0 by *both* p_1 and p_2. The point of the example is to demonstrate how, even though the die landing on an edge is not a Savage-null event, it is "infinitely less likely" than its landing on a face. This terminology is made precise in Sections 5 and 6.

As we mentioned in Section 2, strengthening state independence from Axiom 5 to Axiom 5′ rules out Savage-null events. However, Section 5 shows that in the more general lexicographic framework not all notions of null events are ruled out.

COROLLARY 3.1: *Axioms* 1–3, 4′, *and* 5′ *hold if and only if there is an affine function* u: $\mathscr{P} \to \mathbb{R}$ *and an LPS* $\rho = (p_1, \ldots, p_K)$ *on* Ω *such that, for all* $x, y \in \mathscr{P}^{\Omega}$,

$$x \succcurlyeq y \Leftrightarrow \left(\sum_{\omega \in \Omega} p_k(\omega) u(x_\omega) \right)_{k=1}^{K} \geqslant_L \left(\sum_{\omega \in \Omega} p_k(\omega) u(y_\omega) \right)_{k=1}^{K}.$$

Furthermore, u is unique up to positive affine transformations, the LPS ρ *has the same uniqueness properties as in Theorem* 3.1, *and for each* ω *there is a k such that* $p_k(\omega) > 0$.

The preferences of the decision maker described above for betting on the roll of a die do not satisfy Axiom 5′—landing on a corner is Savage-null. But now suppose the decision maker has third order beliefs

$$p_3(\omega) = \begin{cases} 1/8 & \text{if } s \text{ is a corner,} \\ 0 & \text{otherwise.} \end{cases}$$

As before, face landings are infinitely more likely than edge landings, which in turn are infinitely more likely than corner landings. But now there are no Savage-null events.

4. ADMISSIBILITY AND CONDITIONAL PROBABILITIES

The notion of admissibility and the issues underlying the existence of well defined conditional probabilities are both related to the representation of conditional preferences. In this section we investigate admissibility and prove a result, analogous to Corollary 2.1, on the representation of conditional preferences for non-Archimedean SEU theory.

DEFINITION 4.1: Let u be a utility function. The preference relation $\succcurlyeq$ is *admissible with respect to u* if whenever $u(x_\omega) \geqslant u(y_\omega)$ for all $\omega \in \Omega$, with strict inequality for at least one ω, then $x \succ y$.

This definition is a statement about the behavior of a utility function representing $\succcurlyeq$, rather than a statement about conditional preferences. It is helpful to contrast this definition with Theorem 4.1 below which is stated solely in terms of the conditional preferences. Consider therefore the following class of decision problems. Let $\mathscr{S}$ be a partition of Ω, and for each S in $\mathscr{S}$ let $X(S) \subset \mathscr{P}^S$ be the subset of acts in $\mathscr{P}^S$ that the decision maker can choose among if S occurs. A strategy is then an element of $X \equiv \prod_{S \in \mathscr{S}} X(S) \subset \mathscr{P}^{\Omega}$, and preferences $\succcurlyeq$ are defined on the set of strategies.

THEOREM 4.1: *Suppose* $\succcurlyeq$ *satisfies Axioms* 1 *and* 2. (i) *For* $x, y \in X$, *if* $x \succcurlyeq_S y$ *for all* $S \in \mathscr{S}$ *and* $x \succ_S y$ *for some* $S \in \mathscr{S}$, *then* $x \succ y$. (ii) *For* $x \in X$, *if* $x \succcurlyeq y$ *for all* $y \in X$, *then* $x \succcurlyeq_S z$ *for all* z *such that* $z_S \in X(S)$.

Theorem 4.1 (i) is often referred to as the "sure thing principle." It states that if an act x is conditionally (weakly) preferred to y given any information cell in $\mathscr{S}$, and is strictly preferred given some cell, then x is unconditionally strictly preferred to y. Taking $\mathscr{S}$ to be the finest possible partition leads to a result that clearly resembles admissibility, but whose hypothesis is a claim about conditional preferences rather than utility functions. The second part of Theorem 4.1 states that an optimal strategy must be conditionally optimal on all cells in $\mathscr{S}$, and this bears a resemblance to the logic of backwards induction. The point of this theorem is that both properties are satisfied by Archimedean SEU theory. Admissibility and backwards induction are best understood *not* as conditions on the preferences, but in terms of the *representation* of conditional preferences.[7] Admissibility was defined above in terms of the representation. Similarly, we interpret backwards induction rationality to be the restriction that an optimal strategy maximize conditional expected utility on each cell $S \in \mathscr{S}$; hence well defined conditional probabilities are required.

In this section we suppose henceforth that Axiom 5′ is satisfied. The main purpose is to show that in the non-Archimedean framework this implies admissibility and the existence of LPS's which represent conditional preferences and which are concentrated on the conditioning event.

THEOREM 4.2 (Admissibility): *Suppose* $\succcurlyeq$ *satisfies Axioms* 1–3, 4′, *and* 5′, *and let* u *and* ρ *denote a utility function and an LPS which represent* $\succcurlyeq$. *Then* $\succcurlyeq$ *is admissible with respect to* u.

Theorem 4.2 is an immediate consequence of Corollary 3.1. We now turn to the definition of conditional probabilities for lexicographic hierarchies of beliefs.

DEFINITION 4.2: Let $\rho = (p_1, \ldots, p_K)$ be an LPS on the state space Ω. For any nonempty event S, the *conditional LPS given* S is $\rho_S \equiv (p_{k_1}(\cdot \mid S), \ldots,$

[7] Since the preferences determine the representation, these conditions can be stated in terms of the preferences alone—however as Theorems 4.1–4.3 show, it may be more insightful to think of these properties using the representation.

$p_{k_L}(\cdot|S))$, where the indices k_l are given by $k_0 = 0$, $k_l = \min\{k\colon p_k(S) > 0$ and $k > k_{l-1}\}$ for $l > 0$, and $p_{k_l}(\cdot|S)$ is given by the usual definition of conditional probabilities.

This notion of conditional probability is intuitively appealing—the conditional LPS is obtained by taking conditional probabilities of all p_k's in the LPS ρ for which conditionals are defined ($p_k(S) > 0$) and discarding the other p_k's. Axiom 5′ implies that at least one p_k will not be discarded so that $L \geqslant 1$. In Section 6, where probabilities are allowed to be infinitesimals, it is seen that this definition is equivalent to an exact analog of the usual definition of conditional probabilities. Clearly Definition 4.2 satisfies two of our objectives: it is an LPS (hence a "subjective probability" in the lexicographic framework); and it is concentrated on the conditioning event S. The main issue which remains to be verified is that these conditional probabilities represent conditional preferences, as is shown in Theorem 4.3, which is a non-Archimedean version of Corollary 2.1.

THEOREM 4.3: *Suppose $\succcurlyeq$ satisfies Axioms* 1–3, 4′, *and* 5′, *and let u and ρ denote a utility function and an LPS which represent $\succcurlyeq$. Then for any nonempty event S, the utility function u and the conditional LPS $\rho_S \equiv (p_{k_1}(\cdot|S), \ldots, p_{k_L}(\cdot|S))$ represent the conditional preferences $\succcurlyeq_S$:*

$$x \succcurlyeq_S y \Leftrightarrow \left(\sum_{\omega \in S} p_{k_l}(\omega|S) u(x_\omega) \right)_{l=1}^{L} \geqslant_L \left(\sum_{\omega \in S} p_{k_l}(\omega|S) u(y_\omega) \right)_{l=1}^{L}.$$

5. LEXICOGRAPHIC CONDITIONAL PROBABILITY SYSTEMS

In this section we discuss in more detail the ways in which events can be null in lexicographic probability systems, and examine the relationship between the characterization of Section 3 and other recent developments (Myerson (1986a, b), McLennan (1989a, 1989b), Hammond (1987)). This relationship will be clarified by axiomatizing lexicographic *conditional* probability systems (not to be confused with the conditional LPS's of Definition 4.2!), using an Archimedean property intermediate in strength between Axioms 4 and 4′. This intermediate Archimedean axiom will arise naturally from an understanding of null events in the lexicographic framework.

In the Archimedean framework an event S is infinitely more likely than another event T (in terms of probability ratios) if and only if T is Savage-null. Non-Archimedean theory admits a richer likelihood order on events. We will investigate a partial order on events, $S \gg T$, to be read as "S is infinitely more likely than T." One could proceed to define such a notion in terms of the representation or the preferences. We adopt the latter approach, which provides a more primitive characterization, in order to better understand the relationship with Savage-null events. The following characterization of Savage-null events (Definition 2.2) will be useful.

THEOREM 5.1: *Assume that* $\succcurlyeq$ *satisfies Axioms* 1 *and* 2. *An event* T *is Savage-null if and only if there exists a nonempty disjoint event* S *such that*

$$x \succ_S (\textit{respectively } \sim_S)\ y$$

implies

$$(x_{-T}, w_T) \succ_{S \cup T} (\textit{respectively } \sim_{S \cup T})(y_{-T}, z_T)$$

for all w_T, z_T.

That is, an event T is Savage-null if for some disjoint event S, when comparing (x_S, w_T) and (y_S, z_T), the consequences in the event S are determining for both $\succ_{S \cup T}$ and $\sim_{S \cup T}$. An intuitively weaker order on events arises from supposing that consequences in the event S are determining for $\succ_{S \cup T}$ alone. In the remainder of this section we assume that there are no Savage-null events, and examine the properties of an alternative likelihood ordering on events.

DEFINITION 5.1: For disjoint events $S, T \subset \Omega$ with $S \neq \varnothing$, $S \gg T$ if

$$x \succ_S y \text{ implies } (x_{-T}, w_T) \succ_{S \cup T} (y_{-T}, z_T)$$

for all w_T, z_T.

THEOREM 5.2: *Assume that* $\succcurlyeq$ *is represented by a utility function* u *and an LPS* $\rho = (p_1, \ldots, p_K)$. *For a pair of states* ω^1 *and* ω^2, $\omega^1 \gg \omega^2$ *if and only if* u *and the LPS* $((1,0),(0,1))$ *on* $\{\omega^1, \omega^2\}$ *represent* $\succcurlyeq_{\{\omega^1, \omega^2\}}$.

Theorem 5.2 says that for a pair of *states* the order $\gg$ corresponds to the conditional probabilities (which represent the conditional preferences given that pair of states).[8] More generally, for events, if $S \gg T$ then $p_{k_1}(T|S \cup T) = 0$ (where $k_1 = \min\{k\colon p_k(S \cup T) > 0\}$). However, the converse to this is in general false, so zero probabilities in the representation do not correspond to the ranking $\gg$. A related difficulty with Definition 5.1 is that $S \gg T$ and $S' \gg T$ need not imply $S \cup S' \gg T$. Both these difficulties can be seen in the following example. Consider a state space $\Omega = \{Heads, Tails, Edge, Heads^*\}$ and the LPS $p_1 = (1/2, 1/2, 0)$, $p_2 = (1/2, 0, 1/2)$. Even though $\{Tails\} \gg \{Edge\}$, $\{Heads\} \gg \{Edge\}$, and $p_1(Edge) = 0$, it is not the case that $\{Heads, Tails\} \gg \{Edge\}$ since $x \equiv (2,0,0) \succ (1,1,0) \equiv y$, but $(2,0,0) \prec (1,1,2)$.[9] These problems result from the fact that the supports of p_1 and p_2 overlap. Hence we will now distinguish the subset of LPS's whose component probability measures have disjoint supports, and introduce the Archimedean axiom which is used in Theorem 5.3 below to characterize this subset.

[8] If the strict order $\gg$ is restricted to states, then the induced weak order can be characterized as follows: ω' is not infinitely more likely than $\omega \Leftrightarrow \min\{k\colon p_k(\omega) > 0\} \geqslant \min\{k\colon p_k(\omega') > 0\}$. This weak order turns out to be useful in characterizing proper equilibrium (Myerson (1978))—see the sequel to this paper.

[9] The numbers in these triplets are in utility payoffs.

DEFINITION 5.2: An LPS $\rho = (p_1, \dots, p_K)$, where the supports of the p_k's are disjoint, is a *lexicographic conditional probability system* (LCPS).

AXIOM 4″: *There is a partition* $\{\Pi_1, \dots, \Pi_K\}$ *of* Ω *such that*: (a) *for each* k, *if* $x \succ_{\Pi_k} y \succ_{\Pi_k} z$, *then there exists* $0 < \alpha < \beta < 1$ *such that* $\beta x + (1-\beta)z \succ_{\Pi_k} y \succ_{\Pi_k} \alpha x + (1-\alpha)z$; (b) $\Pi_1 \gg \cdots \gg \Pi_K$.

THEOREM 5.3: *Axioms* 1–3, 4″, *and* 5′ *hold if and only if there is an affine function* u: $\mathscr{P} \to \mathbb{R}$ *and an LCPS* $\rho = (p_1, \dots, p_K)$ *on* Ω *such that, for all* $x, y \in \mathscr{P}^{\Omega}$,

$$x \succcurlyeq y \Leftrightarrow \left(\sum_{\omega \in \Omega} p_k(\omega) u(x_\omega) \right)_{k=1}^{K} \geqslant_L \left(\sum_{\omega \in \Omega} p_k(\omega) u(y_\omega) \right)_{k=1}^{K}.$$

Furthermore, u *is unique up to positive affine transformations*, ρ *is unique*, *and for each* $k = 1, \dots, K$, *the support of* p_k *is* Π_k.

COROLLARY 5.1: *Assume that* $\succcurlyeq$ *is represented by a utility function* u *and an LCPS* $\rho = (p_1, \dots, p_K)$. *For a disjoint nonempty pair of events* S, T,

$$S \gg T \Leftrightarrow k' < l' \text{ for all } k' \in \{k\colon p_k(S|S \cup T) > 0\}$$
$$\text{and } l' \in \{l\colon p_l(T|S \cup T) > 0\}.$$

Theorem 5.3 and Corollary 5.1 show that as a result of strengthening the Archimedean property from Axiom 4′ to 4″, the relation $\gg$ corresponds to zero probabilities in the representation, and Theorem 5.2 can be strengthened to hold for events as well as states.

There is an interesting interpretation of an LCPS. The first-order belief p_1 can be thought of as a prior distribution. If the expected utilities under p_1 of two acts are the same, then the decision maker considers the event $\Omega - \Pi_1$, where by Theorem 5.3, Π_1 is the support of p_1. The second-order belief p_2 is then the "posterior" conditional on the event $\Omega - \Pi_1$. More generally, higher order beliefs can also be thought of as conditional probability distributions.

Return to the example of the coin toss discussed earlier. We now describe how the LPS (p_1, p_2) can be reinterpreted as an LCPS. Suppose there is a possibility that the coin is being tossed in a "dishonest" fashion which guarantees heads. If we denote this event by $\{Heads^*\}$ then the expanded state space is $\Omega^* = \{Heads, \ Tails, \ Edge, \ Heads^*\}$. The beliefs $p_1^* = (1/2, 1/2, 0, 0), p_2^* = (0, 0, 1/2, 1/2)$ on Ω^* have nonoverlapping supports. Moreover, if the events $\{Heads\}$ and $\{Heads^*\}$ are indistinguishable, then (p_1^*, p_2^*) induces the same lexicographic probabilities over the payoff-relevant outcomes—namely, the coin lands on heads, tails, or the edge—as does (p_1, p_2). This example shows how to map an LPS into an LCPS on an expanded state space. A more general treatment is developed in Hammond (1987). However, this reinterpretation does not mean that it suffices to work with LCPS's alone. If in fact {*Heads*} and {*Heads**} are indistinguishable, then bets on {*Heads*} versus {*Heads**} cannot be made and so the subjective probabilities (p_1^*, p_2^*) cannot be derived; including such payoff-irrelevant states contradicts our basic model which admits *all* possible acts in $\mathscr{P}^{\Omega}$.

LCPS's provide a bridge between the work in this paper and some ideas in Myerson (1986a, b). Myerson also starts from the existing SEU theory, but his modification leads in a different direction to that pursued here. Myerson augments the basic preference relation $\succcurlyeq$ by postulating the existence of a distinct preference relation corresponding to each nonempty subset S of the state space Ω. Although interpreted as conditional preferences, these preferences differ from $\succcurlyeq_S$ as defined by Savage (1954) (and Definition 2.1 in this paper). Using this preference structure, Myerson derives the notion of a *conditional probability system* (CPS). The reader is referred to Myerson (1986a, b) for the definition of a CPS, which can be shown to be isomorphic to an LCPS. An important distinction between Myerson's preference structure and ours is that the latter satisfies admissibility (Theorem 4.2) whereas the former does not. McLennan (1989b) uses CPS's to provide an existence proof, and a characterization, of sequential equilibrium.

6. A "NUMERICAL" REPRESENTATION FOR NON-ARCHIMEDEAN SEU

This section provides a "numerical" representation for the preferences described by Axioms 1–3, 4′, and 5′, where the "numbers" are elements in a non-Archimedean ordered field $\mathbb{F}$ which is a strict extension of the real number field $\mathbb{R}$. The field $\mathbb{F}$ is non-Archimedean: it contains both infinite numbers (larger than any real number) and infinitesimal numbers (smaller than any real number) in addition to the reals.[10]

The basic result on the existence of utility functions states that a complete, transitive, and reflexive preference relation on a set X has a real-valued representation if and only if X contains a countable order-dense subset. Without this Archimedean restriction on X, a representation is still possible. A complete, transitive, and reflexive preference relation on *any* set X has a numerical representation taking values in a non-Archimedean ordered field (Richter (1971)). In this paper we have weakened the Archimedean property (Axiom 4) of SEU to Axiom 4′. This weakening still permits a real-valued utility function on consequences, but requires the subjective probability measure to be non-Archimedean. By analogy with the real-valued case, a non-Archimedean probability measure on Ω is a function $p: \Omega \to \mathbb{F}$ such that $p(\omega) \geqslant 0$ for each $\omega \in \Omega$, and $\sum_{\omega \in \Omega} p(\omega) = 1$.

THEOREM 6.1: *Axioms* 1–3, 4′, *and* 5′ *hold if and only if there is an affine function* u: $\mathscr{P}^{\Omega} \to \mathbb{R}$ *and an* $\mathbb{F}$*-valued probability measure* p *on* Ω, *where* $\mathbb{F}$ *is a*

[10] We do not distinguish between the subfield of $\mathbb{F}$ which is order-isomorphic to $\mathbb{R}$ and $\mathbb{R}$ itself. Likewise, $\geqslant$ will be used to denote the order on both $\mathbb{F}$ and $\mathbb{R}$.

non-Archimedean ordered-field extension of $\mathbb{R}$, *such that, for all* $x, y \in \mathscr{P}^{\Omega}$,

$$x \succcurlyeq y \Leftrightarrow \sum_{\omega \in \Omega} p(\omega) u(x_{\omega}) \geqslant \sum_{\omega \in \Omega} p(\omega) u(y_{\omega}).$$

Furthermore, u is unique up to positive affine transformations. If p' *is another* $\mathbb{F}$*-valued probability measure such that u and* p' *represent* $\succcurlyeq$, *then for all* $\omega \in \Omega$, $p(\omega) - p'(\omega)$ *is infinitesimal. Finally,* $p(\omega) > 0$ *for all* $\omega \in \Omega$.

The disadvantage of the representation of Theorem 6.1 is that its proof requires less familiar techniques. Its advantage is that, for most purposes, it really does provide a numerical representation. The probabilities of states can be added, multiplied, divided, and compared just like real numbers. For example, the usual definition of conditional probabilities applies for an $\mathbb{F}$-valued probability measure as well: $p(T|S) \equiv p(T \cap S)/p(S)$ for events $S, T \subset \Omega$ (Theorem 6.1 implies that $p(S) \neq 0$). By analogy with the real-valued case, it is easy to show that

$$x \succcurlyeq_S y \Leftrightarrow \sum_{\omega \in S} p(\omega) u(x_{\omega}) \geqslant \sum_{\omega \in S} p(\omega) u(y_{\omega}).$$

Dividing both sides of the inequality by $p(S)$ shows that u and $p(\cdot|S)$ do indeed represent the conditional preference relation $\succcurlyeq_S$.

We have found that some results are more easily understood by employing the LPS's of Section 3, while others are best seen using the representation of this section. An example of the former is the statement of the uniqueness of the subjective beliefs in the representation theorems. An example of the latter is the issue of stochastic independence discussed in Section 7.

The sufficiency part of Theorem 6.1 can be proved using arguments analogous to those when preferences are Archimedean. The necessity part can be proved in a number of ways, including compactness arguments from logic or by using ultrafilters. A sketch of the ultrafilter argument is provided in the Appendix.[11]

7. STOCHASTIC INDEPENDENCE AND PRODUCT MEASURES

This section considers three possible definitions of stochastic independence. The first requires that the (Archimedean or non-Archimedean) probability measure p be a product measure; the second requires that a decision-theoretic stochastic independence axiom be satisfied; and the third that p be an "approximate product measure." In the Archimedean setting, all three definitions are equivalent, but in the non-Archimedean case the definitions are successively weaker. The notion of independence implicit in refinements of Nash equilibrium such as perfect equilibrium (Selten (1975)) is the first one above. The fact that it is *not* equivalent, in the non-Archimedean setting, to the stochastic

[11] Hammond (1987) has discussed LPS's that can be represented by a probability measure taking values in the non-Archimedean ordered field of rational functions over $\mathbb{R}$.

independence axiom suggests that alternative (weaker) refinements may also be worthy of consideration.[12]

Suppose in the following that the set of states Ω is a product space $\Omega^1 \times \cdots \times \Omega^N$, and for $n = 1, \ldots, N$ let $\Omega^{-n} = \prod_{m \neq n} \Omega^m$. An act x is said to be constant across Ω^n if for every $\omega^n, \tilde{\omega}^n$ in Ω^n and all ω^{-n} in Ω^{-n}, $x_{(\omega^n, \omega^{-n})} = x_{(\tilde{\omega}^n, \omega^{-n})}$. Assume that the preference relation $\succcurlyeq$ satisfies Axioms 1–3 and 5′. We are interested in focusing on the distinction between the Archimedean case in which Axiom 4 holds and the non-Archimedean case in which Axiom 4′ holds.

Definition 7.1: An (Archimedean or non-Archimedean) probability measure p on Ω is a *product measure* if there are probability measures p^n on Ω^n, for $n = 1, \ldots, N$, such that for all $\omega = (\omega^1, \ldots, \omega^N) \in \Omega$, $p(\omega) = p^1(\omega^1) \times \cdots \times p^N(\omega^N)$.

Axiom 6 (Stochastic Independence): *For any $n = 1, \ldots, N$ and every pair of acts x, y that are constant across Ω^n,*

$$x \succcurlyeq_{\{\omega^n\} \times \Omega^{-n}} y \Leftrightarrow x \succcurlyeq_{\{\tilde{\omega}^n\} \times \Omega^{-n}} y.$$

Roughly speaking, Axiom 6 requires that the conditional preferences $\succcurlyeq_{\{\omega^n\} \times \Omega^{-n}}$ and $\succcurlyeq_{\{\tilde{\omega}^n\} \times \Omega^{-n}}$ (viewed as preferences on $\mathscr{P}^{\Omega^{-n}}$) be identical for all ω^n and $\tilde{\omega}^n$ in Ω^n. Suppose the preference relation $\succcurlyeq$ satisfies the Archimedean property (Axiom 4), and let u and p denote a utility function and (Archimedean) probability measure which represent the preferences. It is routine to show that p is a product measure if and only if stochastic independence (Axiom 6) holds. This equivalence breaks down in the non-Archimedean setting. Any preference relation represented by some utility function u and non-Archimedean probability measure p satisfies Axiom 6 if p is a product measure. But the converse is false, as the following example due to Roger Myerson (private communication) shows. Let $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1\} \times \{\omega_1^2, \omega_2^2, \omega_3^2\}$ where the non-Archimedean probability measure is as depicted in Figure 7.1 (in which $\varepsilon > 0$ is an infinitesimal). Fix a utility function u. By Theorem 6.1, the probability measure p and utility function u determine a preference relation which satisfies Axioms 1–3, 4′, and 5′. One can verify that the conditional preference relation given any row is the same and that, likewise, the conditional preference relation given any column is the same. Hence Axiom 6 is satisfied. However, p is not a product measure and, moreover, there is no measure which *is* a product measure and which represents the same preference relation.

In the non-Archimedean case (Axiom 4′ rather than Axiom 4), Axiom 6 is sufficient for the existence of a weaker kind of product measure—a concept known in nonstandard probability theory as S-independence.

[12] Such refinements may involve a sequence of *correlated* trembles converging to a Nash equilibrium—see the sequel to this paper. Related issues are discussed in Binmore (1987, 1988), Dekel and Fudenberg (1990), Fudenberg, Kreps, and Levine (1988), and Kreps and Ramey (1987).

	ω_1^2	ω_2^2	ω_3^2
ω_1^1	$1-2\epsilon-4\epsilon^2-2\epsilon^3-\epsilon^4$	ϵ	$2\epsilon^2$
ω_2^1	ϵ	ϵ^2	ϵ^3
ω_3^1	ϵ^2	ϵ^3	ϵ^4

FIGURE 7.1

DEFINITION 7.2: An (Archimedean or non-Archimedean) probability measure p is an *approximate product measure* if there are probability measures p^n on Ω^n, for $n=1,\ldots,N$, such that for all $\omega=(\omega^1,\ldots,\omega^N)\in\Omega$, $p(\omega)-(p^1(\omega^1)\times\cdots\times p^N(\omega^N))$ is infinitesimal.

Suppose the preference relation $\succcurlyeq$ satisfies Axiom 4′, and let u and p denote a utility function and non-Archimedean probability measure which represent the preferences. It is straightforward to see that if Axiom 6 holds, then p is an approximate probability measure.[13] However, requiring a non-Archimedean probability measure p to be an approximate product measure is strictly weaker than demanding Axiom 6 to hold, as the following example demonstrates. Let $\Omega=\{\omega_1^1,\omega_2^1\}\times\{\omega_1^2,\omega_2^2\}$ where the non-Archimedean probability measure p is as depicted in Figure 7.2 (in which ε and δ are positive infinitesimals with ε/δ infinite). The measure p is an approximate product measure but it is clear that, for any utility function u, the conditional preference relation given the top row differs from that given the bottom row.

[13] Equivalently, in terms of the lexicographic representation of preferences derived in Section 3, Axiom 6 implies that the first-level probability measure p_1 of an LPS $(p_1,\ldots,p_K)$ representing $\succcurlyeq$ is a product measure.

	ω_1^2	ω_2^2
ω_1^1	$1/4 + \delta$	$1/4 + \epsilon$
ω_2^1	$1/4 - \delta$	$1/4 - \epsilon$

FIGURE 7.2

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APPENDIX

This Appendix contains proofs of the results in Sections 3–6. The first two lemmas are used in the proofs of Theorems 3.1 and 6.1.

LEMMA A.1: *Given a preference relation* $\succcurlyeq$ *on* $\mathscr{P}^\Omega$ *satisfying Axioms* 1, 2, 4′, *and* 5′, *there is an affine function* u: $\mathscr{P} \to \mathbf{R}$ *such that for every* $\omega \in \Omega$, $x \succcurlyeq_\omega y$ *if and only if* $u(x_\omega) \geqslant u(y_\omega)$.

PROOF: For each ω, the conditional preference relation $\succcurlyeq_\omega$ satisfies the usual order, independence, and Archimedean axioms of von Neumann-Morgenstern expected utility theory. This follows from the fact that $\succcurlyeq$ satisfies Axioms 1, 2, and 4′, respectively. Hence each $\succcurlyeq$ can be represented by an affine utility function u_ω: $\mathscr{P} \to [0,1]$. Under Axiom 5′, the conditional preference relation $\succcurlyeq_\omega$ is independent of ω, hence every $\succcurlyeq_\omega$ can be represented by a *common* utility function u: $\mathscr{P} \to [0,1]$. *Q.E.D.*

The next step is to use the utility function u to scale acts in utiles: an act $x \in \mathscr{P}^\Omega$ is represented by the tuple $(u(x_\omega))_{\omega \in \Omega} \in [0,1]^\Omega$. The preference relation $\succcurlyeq$ on $\mathscr{P}^\Omega$ induces a preference relation $\geqslant^*$ on $[0,1]^\Omega$: given $a, b \in [0,1]^\Omega$, define $a \geqslant^* b$ if and only if $x \succcurlyeq y$ for some $x, y \in \mathscr{P}^\Omega$ with $u(x_\omega) = a_\omega$, $u(y_\omega) = b_\omega$ for each $\omega \in \Omega$. This definition is meaningful since by Axioms 1 and 2 it is independent of the particular choice of x and y.

LEMMA A.2: *The preference relation* $\geqslant^*$ *on* $[0,1]^\Omega$ *satisfies the order and independence axioms.*

PROOF: This follows immediately from the fact that $\succcurlyeq$ satisfies Axioms 1 and 2. *Q.E.D.*

PROOF OF THEOREM 3.1: Given the preference relation $\succcurlyeq$ on $\mathscr{P}^{\Omega}$, construct the induced preference relation $\geqslant^*$ on $[0,1]^{\Omega}$ of Lemma A.2. By a result of Hausner (1954, Theorem 5.6), there are K (where K is equal to or less than the cardinality of Ω) affine functions U_k: $[0,1]^{\Omega} \to \mathbb{R}$, $k = 1, \ldots, K$, such that for $a, b \in [0,1]^{\Omega}$,

$$a \geqslant^* b \Leftrightarrow (U_k(a))_{k=1}^K \geqslant_L (U_k(b))_{k=1}^K.$$

The next step is to derive subjective probabilities. By linearity, $U_k(a) = \Sigma_{\omega \in \Omega} U_k(e^{\omega}) a_{\omega}$ where e^{ω} is the vector with 1 in the ωth position and 0's elsewhere. By nontriviality (Axiom 3), each U_k can be chosen to satisfy $\Sigma_{\omega \in \Omega} U_k(e^{\omega}) > 0$. So

$$V_k(a) = \sum_{\omega \in \Omega} r_k(\omega) a_{\omega}, \qquad \text{where} \qquad r_k(\omega) = \frac{U_k(e^{\omega})}{\Sigma_{\omega' \in \Omega} U_k(e^{\omega'})},$$

is a positive affine transformation of $U_k(a)$. For each ω, $r_1(\omega) \geqslant 0$ since otherwise $e^{\omega} <^* (0, \ldots, 0)$, contradicting Axiom 3. So we can define a probability measure p_1 on Ω by $p_1 = r_1$. For $k > 1$, find numbers α_i, $i = 1, \ldots, k$, with $\alpha_i \geqslant 0$, $\alpha_k > 0$, and $\Sigma_{i=1}^k \alpha_i = 1$, such that for each ω, $p_k(\omega) = \Sigma_{i=1}^k \alpha_i r_i(\omega) \geqslant 0$. (Again, such α_i's exist since otherwise $e^{\omega} <^* (0, \ldots, 0)$.) The p_k's defined in this way are probability measures on Ω.

To sum up, we have derived probability measures $p_1, \ldots, p_K$ on Ω such that for $a, b \in [0,1]^{\Omega}$,

$$a \geqslant^* b \Leftrightarrow \left(\sum_{\omega \in \Omega} p_k(\omega) a_{\omega} \right)_{k=1}^K \geqslant_L \left(\sum_{\omega \in \Omega} p_k(\omega) b_{\omega} \right)_{k=1}^K.$$

On recalling that $x \succcurlyeq y$ if and only if $a \geqslant^* b$, where $u(x_{\omega}) = a_{\omega}$ and $u(y_{\omega}) = b_{\omega}$ for each ω, the representation of Theorem 3.1 is established. The uniqueness properties and the "if" direction of Theorem 3.1 follow easily from routine arguments. *Q.E.D.*

PROOF OF COROLLARY 3.1: For each $\omega \in \Omega$, there must be a k such that $p_k(\omega) > 0$ since otherwise $\succcurlyeq_{\omega}$ would be trivial, contradicting the fact that under Axiom 5′ there are no Savage-null events. *Q.E.D.*

PROOF OF THEOREM 4.1: (1) It is shown that $x \succcurlyeq_S y$, and $x \succ_T$ (respectively $\succcurlyeq_T$)y for disjoint S and T implies that $x \succ_{S \cup T}$ (respectively $\succcurlyeq_{S \cup T}$) y. A simple induction argument, which is omitted, would complete the proof. By Definition 2.1: $x \succcurlyeq_S y$ implies that $(x_S, x_T, z_{-(S \cup T)}) \succcurlyeq (y_S, x_T, z_{-(S \cup T)})$; and $x \succ_T y$ implies that $(y_S, x_T, z_{-(S \cup T)}) \succ (y_S, y_T, z_{-(S \cup T)})$. By transitivity this implies $(x_S, x_T, z_{-(S \cup T)}) \succ (y_S, y_T, z_{-(S \cup T)})$, which in turn implies $x \succ_{S \cup T} y$. The same argument holds for weak preference, proving the assertion.

(2) If this result is false then there is an event $S \in \mathscr{S}$ and an act z such that $z \succ_S x$ and $z_S \in X(S)$. But then $(z_S, x_{-S}) \in X$ and $(z_S, x_{-S}) \succ x$ by (1) above, so that x is not optimal. *Q.E.D.*

PROOF OF THEOREM 4.2: If $u(x_{\omega}) \geqslant u(y_{\omega})$ for all ω, then for every $k, \Sigma_{\omega \in \Omega} p_k(\omega) u(x_{\omega}) \geqslant \Sigma_{\omega \in \Omega} p_k(\omega) u(y_{\omega})$. Suppose $u(x_{\omega'}) > u(y_{\omega'})$, and let k' be the first k such that $p_k(\omega') > 0$. Then $\Sigma_{\omega \in \Omega} p_{k'}(\omega) u(x_{\omega}) > \Sigma_{\omega \in \Omega} p_{k'}(\omega) u(y_{\omega})$, so $x \succ y$. *Q.E.D.*

PROOF OF THEOREM 4.3: The proof is analogous to that for the SEU case. Clearly $x \succcurlyeq_S y$ if and only if $(x_S, x_{-S}) \succcurlyeq (y_S, x_{-S})$, which in turn is true if and only if

$$\left(\sum_{\omega \in S} p_k(\omega) u(x_{\omega}) + \sum_{\omega \in \Omega - S} p_k(\omega) u(x_{\omega}) \right)_{k=1}^K$$

$$\geqslant_L \left(\sum_{\omega \in S} p_k(\omega) u(y_{\omega}) + \sum_{\omega \in \Omega - S} p_k(\omega) u(x_{\omega}) \right)_{k=1}^K.$$

Subtracting,

$$\left(\sum_{\omega \in S} p_k(\omega) u(x_{\omega}) \right)_{k=1}^K \geqslant_L \left(\sum_{\omega \in S} p_k(\omega) u(y_{\omega}) \right)_{k=1}^K.$$

Finally, for each k, if $p_k(S) > 0$ divide the kth component of both sides by $p_k(S)$. *Q.E.D.*

PROOF OF THEOREM 5.1: *Only if*: Should the displayed equation in Theorem 5.1 fail to hold for any S, then either: $x \succ_S y$ and $(y_{-T}, z_T) \succcurlyeq_{S \cup T} (x_{-T}, w_T)$; or $x \sim_S y$ and $(y_{-T}, z_T) \succ_{S \cup T} (x_{-T}, w_T)$; or $x \sim_S y$ and $(x_{-T}, w_T) \succ_{S \cup T} (y_{-T}, z_T)$. By Theorem 4.1 this implies either $z \succ_T w$, or $w \succ_T z$, so that T is not Savage-null.

If: By Definition 2.2, if T is not Savage-null then there exist w, z such that $w \succ_T z$, but then $x \sim_S x$ while $(x_{-T}, w_T) \succ_{S \cup T} (x_{-T}, z_T)$. *Q.E.D.*

PROOF OF THEOREM 5.2: In an LPS $\rho = (p_1, \ldots, p_K)$, K can be taken to be less than or equal to the cardinality of Ω without loss of generality, so attention can be restricted to LPS's (p_1, p_2). Furthermore, the uniqueness results of Theorem 3.1 imply that for any u, the LPS's $((1,0),(0,1))$ and $((1,0),(\alpha, 1-\alpha))$, for $\alpha < 1$, represent the same preferences. Finally, if $\succcurlyeq_{\{\omega^1, \omega^2\}}$ is represented by $((\beta, 1-\beta),(\gamma, 1-\gamma))$ for some $\beta < 1$ then there exist x, y such that $x \succ_{\omega^1} y$ but $\beta u(x_{\omega^1}) + (1-\beta)u(x_{\omega^2}) < \beta u(y_{\omega^1}) + (1-\beta)u(y_{\omega^2})$, so that $\omega^1 \not\gg \omega^2$. The "if" direction is immediate. *Q.E.D.*

PROOF OF THEOREM 5.3: Since Axiom 4″(a) implies Axiom 4′, it follows from Corollary 3.1 that there is an affine function u: $\mathscr{P} \to \mathbb{R}$ and an LPS $\rho = (p_1, \ldots, p_K)$ such that

$$x \succcurlyeq y \Leftrightarrow \left(\sum_{\omega \in \Omega} p_k(\omega) u(x_\omega) \right)_{k=1}^{K} \geqslant_L \left(\sum_{\omega \in \Omega} p_k(\omega) u(y_\omega) \right)_{k=1}^{K}.$$

It remains to show that ρ can be chosen so that the p_k's have disjoint supports. Let $K_1 = \min\{k: p_k(\Pi_1) > 0\}$. By Axioms 4″(a) and 5′, $p_{K_1}(\omega) > 0$ for all $\omega \in \Pi_1$. By Axiom 4″(a), the p_k's can be chosen so that $p_k(\Pi_1) = 0$ for all $k > K_1$. By Axiom 4″(b), $p_k(\Pi_2) = 0$ for all $k \leqslant K_1$. Next, let $K_2 > K_1$ be defined by $K_2 = \min\{k: p_k(\Pi_2) > 0\}$. Continuing in this fashion shows that ρ can be chosen so that the supports of its component measures are disjoint. The uniqueness properties and the "if" direction follow easily from routine arguments. *Q.E.D.*

PROOF OF COROLLARY 5.1: The proof follows immediately from Theorem 5.3. *Q.E.D.*

PROOF OF THEOREM 6.1: Given the preference relation $\succcurlyeq$ on $\mathscr{P}^\Omega$, construct the induced relation $\geqslant^*$ on $[0,1]^\Omega$ of Lemma A.2. Since $\geqslant^*$ satisfies the order axiom, it follows from general arguments using ultrafilters (see, e.g., Richter (1971, Theorem 9)) that there is a representation for $\geqslant^*$ taking values in a non-Archimedean ordered field $\mathbb{F}$. That is, there exists such a field $\mathbb{F}$ and a utility function U: $[0,1]^\Omega \to \mathbb{F}$ such that for $a, b \in [0,1]^\Omega$,

$$a \geqslant^* b \Leftrightarrow U(a) \geqslant U(b).$$

Furthermore, since $\geqslant^*$ satisfies the von Neumann-Morgenstern independence axiom, it follows from routine separating hyperplane arguments that for every finite set $A \subset [0,1]^\Omega$, there is an affine function U^A: $[0,1] \to \mathbb{R}$ representing $\geqslant^*$ on A. Consequently the ultrafilter argument can be extended to conclude that the utility function U may be taken to be affine. To summarize, we have shown that there is an affine function U: $[0,1]^\Omega \to \mathbb{F}$ representing $\geqslant^*$ on $[0,1]^\Omega$.

The next step is to derive subjective probabilities. By linearity, $U(a) = \sum_{\omega \in \Omega} U(e^\omega) a_\omega$ (where e^ω is the ωth unit vector). By nontriviality (Axiom 3), $\sum_{\omega \in \Omega} U(e^\omega) > 0$. So define

$$V(a) = \sum_{\omega \in \Omega} p(\omega) a_\omega, \qquad \text{where} \qquad p(\omega) = \frac{U(e^\omega)}{\sum_{\omega' \in \Omega} U(e^{\omega'})}.$$

Since V is a positive affine transformation of U, it also represents $\geqslant^*$. The $p(\omega)$'s defined this way constitute an $\mathbb{F}$-valued probability measure on Ω. On recalling that $x \succcurlyeq y$ if and only if $a \geqslant^* b$, where $u(x_\omega) = a_\omega$ and $u(y_\omega) = b_\omega$ for each ω, the representation of Theorem 6.1 is established. *Q.E.D.*

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