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Rationalizable outcomes of large private-value first-price discrete auctions ☆

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Abstract

We consider discrete versions of first-price auctions. We present a condition on beliefs about players' values such that, with any fixed finite set of possible bids and sufficiently many players, only bidding the bid closest from below to one's true value survives iterative deletion of bids that are dominated, where the dominance is evaluated using beliefs that satisfy the condition. The condition holds in an asymmetric conditionally independent environment so long as the likelihood of each type is bounded from below. In particular, with many players, common knowledge of rationality and that all types are possible in an independent and private values auction implies that players will bid just below their true value.

JEL classification: C72; D44

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1. Introduction

We consider first-price auctions with private values and with many players. It is well known that in the unique equilibrium of the symmetric model (with independent values) the bids converge to the true values as the number of bidders is made large and hence the price converges to the highest value. Our analysis here presents a sense in which this

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result is robust to relaxing the solution concept and the assumption that the distribution of types is common knowledge. We assume that the set of valuations and the set of allowable bids are finite and show that in large auctions bidders bid (almost) their true value when it is common knowledge only that players are rational and that the joint distribution of the values satisfies a certain condition. This condition is satisfied, for example, if the distribution of the values is conditionally independent and the likelihood of every value in each state is bounded above zero. Thus, with many bidders (in this discrete environment), the object goes to the bidder with the highest value (efficiency), and almost surely the price is (almost) the highest value, even without imposing the equilibrium assumptions.

Our analysis concerns a special instance of a general issue in auction theory. Since various results on auctions rely on Nash equilibrium as the solution concept, and in addition many of these are sensitive to the specific distribution of values, it is important to investigate the robustness of results to the solution concept and to the assumption of a commonly known distribution of values. In this vein it is often shown in second-price and in ascending auction mechanisms that the Nash equilibria of interest are *ex post* equilibria, i.e., the strategies select best replies against the realized outcomes, so that the results are not sensitive to the distribution of values. However, in first-price auctions such as we analyze here, the literature considers Nash equilibria that are not *ex post* equilibria. Moreover, other than the well-known result that bidding one's value is weakly dominant in private-and-independent value second-price auctions, we know of only two papers in auction theory whose results do not rely on Nash equilibrium.

Chung and Ely (2000) show that in two-person auctions iterated deletion of *ex post* weakly dominated strategies selects the efficient equilibrium of a Vickrey–Clark–Groves auction even when values are interdependent.[1] In a paper very closely related to ours, Battigalli and Siniscalchi (2000) study the implications of common knowledge of rationality (rationalizability) in a first price auction with private independent values. Unlike our model, they adopt the standard (for auction theory) setup of continuum sets of bids and values. They show that any positive bid up to some level strictly above the Nash equilibrium bid is rationalizable. Therefore, in particular, the set of rationalizable strategies in their model does not approach the competitive equilibrium when the number of bidders becomes large.[2] Thus, their result stands in sharp contrast to ours. We will discuss further the difference between these results in the concluding section.

A more distantly related literature explores the eductive justification of the competitive equilibrium. Guesnerie (1992) looks at the set of rationalizable outcomes in a game in which a continuum of suppliers decide simultaneously on the quantities of a homogenous product that they supply and then the price is determined by an exogenously given demand function. He shows that when the supply curve is steeper than the demand curve (in the traditional labeling of price on the vertical axis), then the rationalizable set contains only

[1] A strategy is *ex post* weakly dominated by another if, for every type and action profile of the opponents, the dominated strategy does no better and for some such profile it does strictly worse. We discuss notions of dominance further in Section 4.1 below.

[2] The upper bound does converge to the Nash, hence competitive equilibrium. This follows from the facts that the upper bound cannot be greater than the bidder's value for the object (see also footnote 4 below and the related discussion in the text), and the Nash equilibrium converges to this value.

the competitive equilibrium. One may think of course of the mirror image of that model in which the supply curve is fixed and the buyers decide strategically on their quantities. The corresponding condition in that variation is that the demand curve is steeper than the supply curve. The auction model is not a special case of that variation, since it designates prices rather than quantities as the strategic variables. But, in any case, the condition on the relative slopes does not hold in the auction model, since the supply curve is inelastic at one unit. Thus, the competitive prediction of Guesnerie's model does not apply in the auction model.

We present the model and solution concept in the next section. The results are stated and proven in the following section. The last section discusses the interpretation and further aspects concerning our solution concept and results, and more detail on the relationship of this work to the literature.

2. The model

As mentioned, we consider a first-price auction with private values. Each player $i \in \{1, 2, \ldots, n\}$ is informed of her private value (type), v_i, of the object, and then submits a bid. The object is awarded to the highest bidder who then pays her bid; in the case of ties, the object is awarded with equal probability to one of the tied highest bidders (and only the winner pays the winning bid). We assume that values and bids are on a discrete grid, say $V = \{0, 1/m, 2/m, \ldots, 1 - 1/m, 1\}$, and we denote the size of the grid by $d = 1/m$.

An *ex ante* strategy for player i in this environment, $s_i \in \mathcal{S}_i$, is then a function from i's possible values, V, into the possible bids, V, and a strategy profile $s \in \mathcal{S}$ is an n-tuple of such functions. For our purposes it is more useful to think of *interim* strategies that specify the bid of a player with a particular value. This bid is thus an element of V, and an interim strategy profile is then a $(m + 1) \times n$-tuple specifying what bid each type of each player chooses. Let $u_i(v, b_i, b_{-i})$ denote player i's expected utility when i is of type v, i chooses bid b_i, and i's opponents bid b_{-i}. (Recall that since we assume private values, i's payoffs depend only on i's type.)

We solve the game using iterated deletion of dominated strategies. The version of dominance we use allows the players' beliefs about their opponents' types not to be common knowledge, while at the same time some restriction on these beliefs *is* commonly known.[3] Formally, the conditional beliefs of player i of type v over the types of all other players is a probability measure $p_i(\cdot \mid v_i = v) \in \Delta(V^{n-1})$, where $\Delta(X)$ is the set of probability distributions over the set X. Restrictions on beliefs for a type v are captured by considering only probabilities in subsets denoted by $P_v \subset \Delta(V^{n-1})$. We first define the subset of beliefs to which we restrict attention, and then define the resulting notion of dominance. The relationship between this notion of dominance and other concepts is discussed in the last section.

[3] Formally, what we present here is a "situation" rather than a Bayesian game, since we do not specify commonly-known beliefs. Obviously, we can turn it into a Bayesian game by enriching the sets of possible types, specifying the priors over them and completing it with the assumption that the expanded model is common knowledge. For simplicity we do not take this extra step.

Definition 1. $P_v \subset \Delta(V^{n-1})$ is the collection of subsets of beliefs for each type v satisfying the following two conditions:

Condition 1. *Each player believes with positive probability that he might have the highest valuation: for any $p_i(\cdot \mid v_i = v) \in P_v$,*

$$p_i(v_j < v \; \forall j \neq i \mid v_i = v) > 0 \quad \forall v > 0 \quad \textit{and}$$
$$p_i(v_j = 0 \; \forall j \neq i \mid v_i = 0) > 0. \tag{1}$$

Condition 2. *For sufficiently large n, player i type v assigns a "small" probability to the event that only m or fewer of the bidders have values v as well, conditional on all n having valuations smaller or equal to v:*

There exists N such that, $\forall n > N$, i, and v and any $p_i(\cdot \mid v_i = v) \in P_v$,

$$p_i\big(\#\{j\colon\ v_j = v\} \leqslant m \;\big|\; v_j \leqslant v \; \forall j,\ v_i = v\big) < \frac{1}{n(m-1)+1}. \tag{2}$$

As we show at the end of the next section, if the v_i's are (conditionally) independently distributed and the (conditional) probability of $v_i = 1$ is greater than some $\delta > 0$, for all i, then $p_i(\#\{j\colon\ v_j = v\} \leqslant m \mid v_j \leqslant 1 \; \forall j,\ v_i = 1)$ is bounded by an expression on the order of $n^m(1-\delta)^n$. Therefore, in this case Condition 2 is satisfied since, for large n, $n^m(1-\delta)^n < 1/(n(m-1)+1)$.

Let $\mathcal{P}$ denote the collection of the sets P_v, for each possible type, i.e., $\mathcal{P} \triangleq (P_v)_{v \in V} \subset [\Delta(V^{n-1})]^V$.

Definition 2. The bid b_i is $\mathcal{P}$-dominated for type v of player i by b_i' given that opponents' strategies are restricted to $S_{-i} \subset \mathcal{S}_{-i} \triangleq \{s_{-i} : V^{n-1} \to V^{n-1}\}$ if for all $p_i(\cdot \mid v_i = v) \in P_v$ and all $s_{-i} \in S_{-i}$,

$$\sum_{v_{-i} \in V^{n-1}} p_i(v_{-i} \mid v_i = v) u_i\big(v, b_i', s_{-i}(v_{-i})\big)$$
$$> \sum_{v_{-i} \in V^{n-1}} p_i(v_{-i} \mid v_i = v) u_i\big(v, b_i, s_{-i}(v_{-i})\big).$$

When i's type and the set to which opponents' strategies are restricted is obvious we will simply say that b_i is *$\mathcal{P}$-dominated by b_i'*, and if it is dominated by some $b_i' \in V$ we will just say that it is *$\mathcal{P}$-dominated*.

3. The results

3.1. Players bid the highest bid below their value

Our first result is that the only bid that survives iterated deletion of $\mathcal{P}$-dominated bids is $v - d$. That is, each bidder bids the highest price that is still below her valuation. We then prove that beliefs are in $\mathcal{P}$ when bidders' types are drawn from a conditionally independent and symmetric distribution in which the probability of each type is bounded away from zero.

Proposition 1. *There exists N such that, for all $n > N$, the bid $v - d$ is the only bid for a player of type $v > 0$ that survives iterated elimination of $\mathcal{P}$-dominated bids.*

The intuition for this result is as follows. First, we observe that bidders with positive valuations will bid strictly below their valuations. This follows from Condition 1 and iterated deletion of bids at or above a bidder's own value (starting from those with $v = 1$ and proceeding inductively to those with lower valuations). Second, we observe that, for sufficiently large n, the bid $v - d$ dominates all lower bids for a type v. Consider the type $v = 1$ and assume that some bid $b < 1 - d$ is the lowest bid that survived iterated $\mathcal{P}$-dominance for any player with this type. Bidding b is clearly not best if other players of type $v = 1$ are around and are bidding more than b. It is also not best if there are many other players of type $v = 1$ who are bidding b. It may be best otherwise, that is, if there are few enough players of type 1 and they all bid b. We show that, for n large enough, Condition 2 implies that the loss in expected payoff from bidding $1 - d$ instead of b in the otherwise event is smaller than the gain in expected payoff from bidding $1 - d$ instead of b in the preceding two events.

Proof. We iteratively delete strategies that are dominated, where in each iteration we consider a situation that remains after the preceding dominated strategies have been deleted. For any bidder i, bidding 1 is dominated by bidding 0 for all types $v_i < 1$ since a bid of 1 may win, and then such a type will end up with a negative payoff.[4] Next, bidding 1 is dominated by bidding $1 - d$ for $v_i = 1$, because bidding 1 yields a payoff of 0 and, by Condition 1 and the previous step, bidding $1 - d$ can yield a positive payoff. Now bidding $1 - d$ is dominated by bidding zero for all types $v_i < 1 - d$, and therefore bidding $1 - d$ is dominated by bidding $1 - 2d$ for $v_i = 1 - d$. Iterating we conclude that it is dominated for any type v_i of any bidder i to bid more than $v_i - d$, except type zero who bids zero. Notice that the foregoing argument uses (informally) only the assumption that Condition 1 is common knowledge.

Let b_n be the lowest bid that survives iterated deletion of $\mathcal{P}$-dominated bids, for any bidder with type $v = 1$, when there are n bidders. We now argue that for n large enough $b_n = 1 - d$. Assume to the contrary that $b_n < 1 - d$. We show that, for large n, the bid $1 - d$ $\mathcal{P}$-dominates b_n, for each bidder, in contradiction to the definition of b_n.

Consider some bidder i, a distribution $p_i(v_{-i} \mid v_i = v) \in P_v$, and a collection of strategies $s_{-i} : V^{n-1} \to V^{n-1}$ that survive iterated elimination of $\mathcal{P}$-dominated bids (more precisely, strategies such that if $v_{-i} = s_{-i}(\hat{v}_{-i})$ then every element of v_{-i} survived the iterated deletion procedure). In particular, for v_{-i} in which $v_j = 1$ for some j, the jth element of $s_{-i}(v_{-i})$ contains only bids greater than or equal to b_n. For these

[4] Bidding more than v is not necessarily dominated since one can believe that all types are bidding even more, so that one gets a payoff of zero in any case.

$p_i(v_{-i} \mid v_i = v)$ and s_{-i}'s, let $q(k \mid \ell)$ denote the probability that k bidders other than i with values $v = 1$ bid b_n, conditional on there being $\ell \geqslant k$ bidders other than i of type $v = 1$. The profit to bidder i with $v_i = 1$ from bidding $1 - d$ is at least

$$L \triangleq d \times \Biggl(p_i(v_j < 1, \forall j \neq i \mid v_i = 1) + \sum_{\ell=1}^{n-1} p_i\bigl(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\bigr) \sum_{k=0}^{\ell} q(k \mid \ell) \frac{1}{\ell - k + 1} \Biggr). \tag{3}$$

This is the benefit from winning with bid $1 - d$ times a lower bound on the probability of winning with this bid. The probability of winning (conditional on $v_i = 1$) is at least the probability of everyone else having value $v < 1$ plus a lower bound on the probability of winning in the event that there are some players with type $v = 1$. The latter bound is a sum of probabilities of there being ℓ players with type $v = 1$ times the probability $q(k \mid \ell)$ that k of those players bid b_n times the probability of winning if the remaining $\ell - k$ are also bidding $1 - d$. This is a lower bound since some of those $\ell - k$ players who bid above b_n may still bid below $1 - d$.

The profit from bidding b_n is at most

$$U \triangleq (1 - b_n) \times \Biggl(p_i(v_j < 1,\ \forall j \neq i \mid v_i = 1) + \sum_{\ell=1}^{n-1} p_i\bigl(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\bigr) q(\ell \mid \ell) \frac{1}{\ell + 1} \Biggr). \tag{4}$$

Again this is the benefit of winning times an upper bound on the probability of winning. The probability of winning is at most the probability that everyone else has value $v < 1$ plus the probability of there being ℓ players with type $v = 1$ times the probability $q(\ell \mid \ell)$ that all those players bid b_n, divided by $\ell + 1$ and summed over all possible values of ℓ. This is an upper bound because even when everyone has value $v < 1$, they may bid more than b_n.

We want to argue that $L > U$ for large n. To this end, we partition the summations in (3) and (4) into ℓ's that are no more than m, and those that are greater than m, and weaken the bounds further. First, since

$$q(\ell \mid \ell) \frac{1}{\ell + 1} \leqslant \sum_{k=0}^{\ell} q(k \mid \ell) \frac{1}{\ell - k + 1},$$

we have

$$d \times \Biggl(p_i(v_j < 1,\ \forall j \neq i \mid v_i = 1) + \sum_{\ell=1}^{m} p_i\bigl(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\bigr) \sum_{k=0}^{\ell} q(k \mid \ell) \frac{1}{\ell - k + 1} \Biggr)$$

$$\geqslant d \times \Bigg(p_i(v_j < 1, \forall j \neq i \mid v_i = 1) + \sum_{\ell=1}^{m} p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\big) q(\ell \mid \ell) \frac{1}{\ell+1} \Bigg) \triangleq L_1. \tag{5}$$

Second, since

$$q(\ell \mid \ell) + \big(1 - q(\ell \mid \ell)\big)\frac{1}{\ell+1} \leqslant \sum_{k=0}^{\ell} q(k \mid \ell) \frac{1}{\ell - k + 1},$$

$$d\Bigg(\sum_{\ell=m+1}^{n-1} p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\big) \sum_{k=0}^{\ell} q(k \mid \ell) \frac{1}{\ell - k + 1} \Bigg)$$
$$\geqslant d\Bigg(\sum_{\ell=m+1}^{n-1} p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\big) \times \bigg(q(\ell \mid \ell) + \big(1 - q(\ell \mid \ell)\big)\frac{1}{\ell+1} \bigg) \Bigg) \triangleq L_2.$$

Define

$$U_1 \triangleq (1 - b_n) \times \Bigg(p_i(v_j < 1,\ \forall j \neq i \mid v_i = 1) + \sum_{\ell=1}^{m} p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\big) q(\ell \mid \ell) \frac{1}{\ell+1} \Bigg), \tag{6}$$

and

$$U_2 \triangleq (1 - b_n) \times \Bigg(\sum_{\ell=m+1}^{n-1} p_i\big(\#\{j \neq i \text{ and } v_j = 1\} = \ell \bigm| v_i = 1\big) q(\ell \mid \ell) \frac{1}{\ell+1} \Bigg).$$

Clearly $L - U \geqslant (L_1 - U_1) + (L_2 - U_2)$. Observe from (5) and (6) that

$$L_1 - U_1 = (-1 + b_n + d) \times \Bigg(p_i\big(v_j < 1,\ \forall j \neq i \mid v_i = 1\big) + \sum_{\ell=1}^{m} p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\big) q(\ell \mid \ell) \frac{1}{\ell+1} \Bigg)$$
$$\geqslant -(1 - d) \times p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} \leqslant m \bigm| v_i = 1\big). \tag{7}$$

Since $b_n < 1 - d$ we have $L_1 - U_1 < 0$. On the other hand, we now show that $L_2 - U_2 > 0$.

$$L_2 - U_2 = \sum_{\ell=m+1}^{n-1} p_i\big(\#\{j \neq i \text{ s.t. } v_j = 1\} = \ell \bigm| v_i = 1\big)$$

$$\times\left(d\left(q(\ell\mid\ell)+\frac{1-q(\ell\mid\ell)}{\ell+1}\right)-(1-b_n)\frac{q(\ell\mid\ell)}{\ell+1}\right)$$
$$=\sum_{\ell=m+1}^{n-1}p_i\big(\#\{j\neq i\text{ s.t. }v_j=1\}=\ell\;\big|\;v_i=1\big)\frac{1}{\ell+1}$$
$$\times\left(d+\ell q(\ell\mid\ell)\left(d-\frac{1-b_n}{\ell}\right)\right).$$

Since $\ell > m$, we have $d-(1-b_n)/\ell > 0$. Therefore, $L_2-U_2>0$ and L_2-U_2 is minimized when $q(\ell\mid\ell)=0$. Since $\ell<n$, we also have

$$L_2-U_2>\frac{d}{n}\big(1-p_i\big(\#\{j\neq i\text{ s.t. }v_j=1\}\leqslant m\;\big|\;v_i=1\big)\big)>0. \tag{8}$$

We want to show that, if $b_n<1-d$, then bidder i with $v_i=1$ would prefer bidding $1-d$ to b_n, i.e., that $L_2-U_2>-(L_1-U_1)$. From Condition 2

$$p_i\big(\#\{j\neq i\text{ s.t. }v_j=1\}\leqslant m\;\big|\;v_i=1\big)<\frac{1}{n(m-1)+1},$$

and since $(1-d)=(m-1)d$, it follows from (7) and (8) that $L_2-U_2>-(L_1-U_1)$.

We have therefore shown that, for n large enough, the following holds. For any i, any $p_i(v_{-i}\mid v_i=1)\in P_1$ and any strategies s_{-i}, which only prescribe bids that survived iterated elimination of $\mathcal{P}$-dominated bids, we have

$$\sum_{v_{-i}\in V^{n-1}}p_i(v_{-i}\mid v_i=1)u_i\big(1,b_n,s_{-i}(v_{-i})\big)$$
$$<\sum_{v_{-i}\in V^{n-1}}p_i(v_{-i}\mid v_i=1)u_i\big(1,1-d,s_{-i}(v_{-i})\big).$$

That is, the bid $1-d$ $\mathcal{P}$-dominates b_n contrary to the supposition. Therefore, for n large enough, the minimal bid that survives the iterated elimination procedure for any bidder with $v=1$ is $1-d$.

Consider next type $v=1-d$. Since for this type, only bids less than $1-d$ survive iterated elimination, this type only wins if no players are of type $v=1$. Hence, their bidding behavior can be analyzed conditional on there being no players of type $v=1$. But then the analysis above implies that, for n large enough, the only bid that survives iterated deletion of $\mathcal{P}$-dominated bids is $v-2d$. Continuing in this way shows that iterated deletion yields the outcome described in the proposition. □

3.2. *A version of the standard i.i.d. private-values model satisfies Conditions* 1 *and* 2

We now describe a familiar environment in which the beliefs belong to the collection $\mathcal{P}$ defined in Definition 1. Consider the above auction environment with the following special features: the bidders are symmetric; the bidders' types are conditionally independent; and the probability of each type in each state is bounded away from zero. The following proposition establishes that the beliefs in the Bayesian game that describes this case belong to the collection $\mathcal{P}$.

Proposition 2. *Suppose that there are k states of nature $\theta_1, \ldots, \theta_k$ occurring with probabilities $\sigma_1, \ldots, \sigma_k$ and that conditional on θ_i the valuations of the bidders are symmetrically distributed according to i.i.d. random variables such that* $\Pr(v_i = v \mid \theta_j) \geqslant \delta > 0$ *for all v, i and j. The beliefs in the Bayesian game that describes this case satisfy Conditions* 1 *and* 2.

Proof. Condition 1 is clearly satisfied. Considering then Condition 2, let $\gamma_j = \Pr(v_i = 1 \mid \theta_j)$ and observe that in this case

$$\begin{aligned} &p_i\bigl(\#\{j \neq i \text{ s.t. } v_j = 1\} \leqslant m \bigm| v_i = 1\bigr) \\ &\quad = \sum_{j=1}^{k} \Pr(\theta_j \mid v_i = 1)\left((1-\gamma_j)^{n-1} + \sum_{\ell=1}^{m}\binom{n-1}{\ell}(1-\gamma_j)^{n-1-\ell}\gamma_j^{\ell}\right) \\ &\quad = \sum_{j=1}^{k} \frac{\gamma_j \sigma_j}{\gamma_1\sigma_1 + \cdots + \gamma_k\sigma_k}\left((1-\gamma_j)^{n-1} + \sum_{\ell=1}^{m}\binom{n-1}{\ell}(1-\gamma_j)^{n-1-\ell}\gamma_j^{\ell}\right). \end{aligned} \tag{9}$$

Each one of the bracketed terms is bounded as follows

$$\begin{aligned} (1-\gamma_j)^{n-1} + \sum_{\ell=1}^{m}\binom{n-1}{\ell}(1-\gamma_j)^{n-1-\ell}\gamma_j^{\ell} &< (m+1)n^m(1-\gamma_j)^{n-m-1} \\ &\leqslant (m+1)n^m(1-\delta)^{n-m-1}. \end{aligned} \tag{10}$$

Observe that, for sufficiently large n,

$$(m+1)n^m(1-\delta)^{n-m-1} < \frac{1}{n(m-1)+1}. \tag{11}$$

This can be verified by multiplying both sides by $n(m-1)+1$, writing $(1-\delta)^{n-m-1}$ as $1/(1/(1-\delta))^{n-m-1}$ and applying L'Hopital rule repeatedly $m+1$ times to this expression to conclude that the left-hand side after multiplication converges to zero as n grows.

Now (9)–(11) together imply that there is a level N such that, for all $n > N$,

$$p_i\bigl(\#\{j \neq i \text{ s.t. } v_j = 1\} \leqslant m \bigm| v_i = 1\bigr) < \frac{1}{n(m-1)+1}.$$

Essentially the same argument is used to establish

$$p_i\bigl(\#\{j\colon\ v_j = v\} \leqslant m \bigm| v_j \leqslant v\ \forall j, v_i = v\bigr) < \frac{1}{n(m-1)+1},$$

for any v. It follows that, in the symmetric model, if the p_i's are conditionally independent with full support in the sense described above, then $p_i(\cdot \mid v_i = v) \in P_v$. □

4. Discussion

4.1. The solution concept

The solution concept employed above is iterated deletion of $\mathcal{P}$-dominated strategies. In the following discussion we relate it to other notions of dominance in general games of incomplete information. We also relate this to Battigalli's (1999) notions of rationalizability in such games, and use this to argue that common knowledge of rationality and of the fact that the beliefs belong to $\mathcal{P}$ imply that only bids that survive iterated deletion of $\mathcal{P}$-dominated strategies will be used.

Clearly the definition of $\mathcal{P}$-domination applies to any restriction on beliefs, not only to the particular set $\mathcal{P}$ we defined. To present the general version of this definition, consider a game of incomplete information with player set I, type spaces T_i for each player i, action spaces A_i for each player i, and utility functions, $u_i : A \times T \to \Re$. As usual, $t \in T$ and $t_{-i} \in T_{-i}$ are, respectively, profiles of types for all players and for players other than i; the same convention is used for $a \in A$ and $a_{-i} \in A_{-i}$. As before, let $S_{-i} \subset \mathcal{S}_{-i} \triangleq \{s_{-i}\colon T_{-i} \to A_{-i}\}$ be a subset of strategies for i's opponents. Denote mixed actions for i by $\alpha_i \in \Delta(A_i)$, and let $P_{t_i} \subset \Delta(T_{-i})$ be a subset of player i's possible beliefs as type t_i. (As is standard we extend the utility function to mixed strategies using linearity, writing $u_i(\alpha_i, a_{-i}, t)$ for i's expected utility from playing α_i against a_{-i} when types are t.) The definition below extends our earlier definition to general games with any, not necessarily symmetric, restrictions on players' beliefs.

Definition 3. The action a_i is $P_{\bar{t}_i}$-dominated for $\bar{t}_i$ by α_i, given that opponents' strategies are restricted to S_{-i}, if for all $p_i(\cdot \mid t_i = \bar{t}_i) \in P_{\bar{t}_i}$ and all $s_{-i} \in S_{-i}$,

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i = \bar{t}_i) u_i\big(\alpha_i, s_{-i}(t_{-i}), t\big) > \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i = \bar{t}_i) u_i\big(a_i, s_{-i}(t_{-i}), t\big).$$

In this general definition the domination can be by *mixed* actions, whereas Definition 2 in Section 2 admits only domination by pure actions. While domination via mixed actions is clearly the appropriate concept, the weaker notion of Definition 2 is both somewhat simpler and sufficient for our main result.

Remark 1 (Dominance and never a best reply). While we define our solution concept in terms of dominated strategies, we could equivalently define it in terms of iterative deletion of strategies that are never best replies to any beliefs about opponents and any beliefs satisfying condition $\mathcal{P}$. Formally, the action a_i is never a $P_{\bar{t}_i}$-best reply for $\bar{t}_i$, given that opponents' strategies are restricted to S_{-i}, if for all $p_i(\cdot \mid t_i = \bar{t}_i) \in P_{\bar{t}_i}$ and all $\sigma_{-i} \in \Delta(S_{-i})$ there exists $a_i'(p_i, \sigma_{-i})$ s.t.

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i = \bar{t}_i) u_i\big(a_i'(p_i, \sigma_{-i}), \sigma_{-i}(t_{-i}), t\big)$$
$$> \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i = \bar{t}_i) u_i\big(a_i, \sigma_{-i}(t_{-i}), t\big).$$

If $P_{\bar{t}_i}$ is convex then a_i is never a $P_{\bar{t}_i}$-best reply for $\bar{t}_i$ if and only if it is $P_{\bar{t}_i}$-dominated for $\bar{t}_i$. To see this consider the agent game where each type is a player, and Nature is a player choosing which "type" will get to play. The equivalence then follows from the usual arguments (see Pearce (1984, Lemma 3), van Damme (1987, Lemma 3.2.1) or Myerson (1991, Theorem 1.6)) so long as $P_{\bar{t}_i}$ is convex. Note that when $P_{\bar{t}_i}$ is not convex, never best replies may be undominated. (This can be seen in the game in Remark 3 below, but with

P_{row} containing the two extreme beliefs that the column player is either the left type or the right type for sure. In this situation D is undominated but it is never a best reply—either U or M is better, depending on the belief in P_{row}.)

Using the above equivalence it is easy to see that our solution concept is the same as (a static, correlated, n-person version of) Battigalli's (1999) notion of weak (and strong) Δ-rationalizability (where Δ is the counterpart of our P_{t_i}). Battigalli argues that the Δ-rationalizable set is the set implied by common knowledge of rationality and of the beliefs satisfying Δ. This then means that the actions surviving iterated deletion of P_{t_i}-dominated strategies are those corresponding to common knowledge of rationality and of the beliefs being contained in $\mathcal{P}$.

We can now rephrase the main result in terms of this interpretation.

Corollary 1. *For sufficiently large n only the strategy profile of bidding just below one's value is consistent with common knowledge of rationality and that beliefs are in $\mathcal{P}$.*

Remark 2 (Interim and ex post dominance). If P_{t_i} is a singleton, say p_i, then a_i is P_{t_i}-dominated if and only if it is *interim dominated*. At the other extreme, if $T_{-i} \subset P_{t_i}$ (where we abuse notation by writing t_{-i} for the measure in $\Delta(T_{-i})$ that assigns probability one to the point t_{-i}) then a_i is P_{t_i}-dominated if and only if a_i is *ex post* dominated.[5] (This follows from the immediate observation that a_i is P_{t_i}-dominated if and only if it is $\text{co}(P_{t_i})$-dominated, where $\text{co}(P_{t_i})$ denotes the convex hull of P_{t_i}.) Thus, P_{t_i}-dominance is intermediate between *ex post* dominance and *interim* dominance. Moreover, using the interpretation discussed above, it also follows that iterated deletion of *ex post* dominated strategies corresponds to common knowledge of rationality (with no restrictions whatsoever on beliefs). This is the obvious analog to the characterization of iterated deletion of *interim* dominated strategies in a game of incomplete information with *given* beliefs p_i as equivalent to common knowledge of rationality and of the game, hence also of those beliefs.

Remark 3 (Private values). A game with private values is such that $u_i(a_i, s_{-i}(t_{-i}), t)$ depends directly only on t_i rather than the entire vector of types t. In games with private values, if $S_{-i} = \mathcal{S}_{-i}$, so that all possible opponents' strategies are allowed, then the set of P_{t_i}-dominated strategies is the same for all P_{t_i}. In particular, the set of (un)dominated strategies is the same for *ex post* and *interim* dominance. However, in subsequent rounds of iterated deletion $S_{-i} \subsetneq \mathcal{S}_{-i}$, and this independence of P_{t_i} is no longer true in general.

To see this consider the private-values game below, in which the column player has two types. After deleting dominated strategies for the column player, the action U is P-dominated if and only if all $p \in P$ assign the left type of the column player probability less than $2/3$.[6]

[5] As mentioned, Chung and Ely (2000) analyze iterated deletion of strategies that are weakly *ex post* dominated in an auction context.

[6] If we interpret the two games as different types of the *row* player then this is like the example used by Fudenberg and Tirole (1991, p. 229) to demonstrate the relationship between *ex ante* and *interim* dominance: UM is *ex ante* dominated but not *interim* dominated for the belief that assigns equal probability to both types of the row player.

	L	R		L	R
U	3, 1	0, 0	U	3, 0	0, 1
M	0, 1	3, 0	M	0, 0	3, 1
D	2, 1	2, 0	D	2, 0	2, 1

Remark 4 (Correlation). A form of correlation, or communication, is implicit in the definition above. It allows player i to believe that the strategy of player j can depend on the type of player k. If one requires that $S_{-i} = \prod_{j\neq i} S_j$, so that such correlation is prohibited, then, in general, more strategies are dominated (since they need be worse against a smaller set—those that are not correlated in this manner—of opponents' strategies). Nevertheless, there are two conditions under which it is irrelevant whether or not one allows for this form of correlation. If we consider *ex post* dominance ($T_{-i} \subset P_{t_i}$) then it is clearly irrelevant. It is slightly less obvious and more interesting to observe that this restriction is also irrelevant in games with private values. We did not impose this restriction above as it would not simplify the proof or notation.

To see why this restriction is irrelevant in private-values games, argue by contradiction. Assume that a_i is P_{t_i}-dominated by α_i when this correlation is prohibited, so that $\sum_{t_{-i}\in T_{-i}} p_i(t_{-i} \mid t_i = \bar{t}_i)u_i(\alpha_i, s_{-i}(t_{-i}), \bar{t}_i) > \sum_{t_{-i}\in T_{-i}} p_i(t_{-i} \mid t_i = \bar{t}_i)u_i(a_i, s_{-i}(t_{-i}), \bar{t}_i)$ for all $s_{-i} \subset \prod_{j\neq i} S_j$ and all $p_i(t_{-i} \mid t_i = \bar{t}_i) \in P_{\bar{t}_i}$, and that a_i is not $P_{\bar{t}_i}$-dominated by α_i when this correlation is permitted, so that

$$\sum_{t_{-i}\in T_{-i}} p_i^*(t_{-i} \mid t_i = \bar{t}_i)u_i\big(\alpha_i, s_{-i}^*(t_{-i}), \bar{t}_i\big) \leqslant \sum_{t_{-i}\in T_{-i}} p_i^*(t_{-i} \mid t_i = \bar{t}_i)u_i\big(a_i, s_{-i}^*(t_{-i}), \bar{t}_i\big)$$

for some $s_{-i}^*: T_{-i} \to A_{-i}$, $s_{-i}^* \notin \prod_{j\neq i} S_j$, and some $p_i^*(t_{-i} \mid t_i = \bar{t}_i) \in P_{\bar{t}_i}$. Therefore, $u_i(\alpha_i, s_{-i}^*(t_{-i}^*), \bar{t}_i) \leqslant u_i(a_i, s_{-i}^*(t_{-i}^*), \bar{t}_i)$ for some t_{-i}^*, so for $s_{-i} = s_{-i}^*(t_{-i}^*) \in \prod_{j\neq i} S_j$ the first inequality is not satisfied.

4.2. *Alternative sufficient conditions*

Proposition 2 shows that beliefs in a symmetric model with (conditionally) independent values whose likelihoods are bounded away from zero satisfy Conditions 1 and 2. It is easy to see that the result holds as stated also when there are infinitely many states of nature. Furthermore, it is clear that in the finite-state case the bounds on the likelihoods of the values are not needed for the conclusion that, in the limit, almost surely, the winning bid is $1 - d$.[7] Furthermore, symmetry does not play an important role in the proof of Proposition 2: an asymmetric model of conditional independence that assumes $\Pr(v_i = v \mid \theta_j) \triangleq \gamma_{j,i,v} \geqslant \delta > 0$ for all v, i and j, would generate the same result.[8] Thus, the assumption of our general model that it is commonly known that the bidders' beliefs belong to the set $\mathcal{P}$, holds in a situation in which it is commonly known that the underlying structure satisfies conditional independence and the δ-full-support requirement.

[7] Let $\delta = \min\{\gamma_k\colon \gamma_k > 0\}$, and note that for j such that $\gamma_j = 0$ the terms in the summation are zero.

[8] The only difference is that in Eqs. (9)–(11) expressions like $(1 - \gamma_j)^h$ and γ_j^h will be replaced by products like $(1 - \gamma_{j,i_1,v}) \times (1 - \gamma_{j,i_2,v}) \times \cdots \times (1 - \gamma_{j,i_h,v})$ and $\gamma_{j,i_1,v} \times \gamma_{j,i_2,v} \times \cdots \times \gamma_{j,i_h,v}$.

Can the model be generalized further? The preceding discussion suggests the following conjecture: if the values of the players are exchangeable and $\Pr(v_i \mid v_j)$ is bounded away from 0, for all i and j, then Conditions 1 and 2. The basis for this conjecture is de Finetti's Theorem. It implies that, if the values of the players are exchangeable, then their joint distribution is as that of a collection of conditionally independent and symmetric random variables like those described in the hypothesis of Proposition 2 (except that in general there will be infinitely many conditioning θ's). However, to invoke Proposition 2, it is also required that $\Pr(v_j \mid \theta)$ be bounded away from 0, for all θ, since this property is used in the proof. But the fact that $\Pr(v_j \mid v_i)$ is bounded away from 0 does not imply the boundedness of $\Pr(v_j \mid \theta)$ for all θ, and we have not been able to modify the proof of Proposition 2 in a manner that circumvents this problem.

4.3. The number of iterations and of players needed

Since iterated dominance arguments may appear to be stronger as they rely on fewer iterations, it is natural to comment on the number of iterations needed for our main result. The first step of the iterative deletion process shows that a type v player will bid at most $v-d$; this step requires as many iterations as the number of possible bids minus one. (Note that only one iteration is needed to conclude that type v players bid at most $v-d$ if *weakly* dominated strategies are deleted.) The second step of the deletion process shows that for n large enough, bidding below $v-d$ is dominated by $v-d$. The number of iteration used in the proof above to establish this step equals the number of possible bids.[9] However, only one iteration is needed to show that, in the limit, almost surely, the winning bid is $1-d$.

Thus the number of iterations required for the result is roughly linear in the number of possible bids, and this can be reduced to two if the solution concept is strengthened to be based on weak dominance and the conclusion is weakened to hold with high probability for the winning bid.

It is also worth noting that the bounds that the propositions yield regarding the number of players needed are loose. For instance, consider the standard independent private-value model with a uniform distribution. Then for Eq. (11) to be satisfied when $m=2$, approximately $n>30$ is necessary. However, it is easy to see that $n=2$ implies that a player with value 1 bids $1/2$.

4.4. Finiteness

A key assumption for our results is the finiteness of the set of possible bids. To understand the role of finiteness, consider the case where bids must be in $B=\{1/i\colon i=1,2,\ldots\}$, and let the values be distributed uniformly on the unit interval. In this case it is easy to see that for *any* m large enough, the bid $d=1/m$ survives iterated deletion of $\mathcal{P}$-dominated bids for all types with $v>1/(m-1)$. (The bid d is a best reply to the strategy profile in which everyone with $v>1/m$ bids $1/(m+1)$, and so on, so

[9] The proof does not delete as many strategies as possible in each step so the result may require fewer iterations.

survives iterative deletion.) As another example, observe that in the symmetric model with independent values and a continuum of types, $\bar{v}$ bidding half the Nash equilibrium bid is not iteratively dominated. (This strategy is a best reply to types below $\bar{v}$ bidding half their Nash equilibrium bids, and those above bidding their Nash equilibrium bids, so will never be deleted.)

Battigalli and Siniscalchi (2000) analyze the case where the bids and values are not on a grid (thus are any number in $[0, 1]$) and allow for any n (not necessarily large). As follows from the above examples, they show that any small positive bid is rationalizable. They also go beyond this intuition and show that the rationalizable set includes any bid between 0 and some bid that is strictly greater than the Nash equilibrium bid, and they provide methods for calculating the upper bound precisely.

Thus, the finiteness of the possible bids is crucial. However, the finiteness of the type space does not seem crucial. It seems obvious, though we have not verified all the details, that our analysis carries through also when only the bids are restricted to a finite grid, and it is commonly known that the values are distributed according to some distribution function with density at least δ on [0,1]. The result would then be that for any m, $\eta \in (0, 1/m)$, and $\delta > 0$ there exists $N(m, \eta, \delta)$such that for any $n > N(m, \eta, \delta)$ only the bid k/m will survive iterated deletion of P-dominated strategies for any type $v \in [k/m + \eta, (k+1)/m]$.

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