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Journal of Economic Theory 121 (2005) 259–272

JOURNAL OF  
**Economic  
Theory**

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# Reputation with equal discounting in repeated games with strictly conflicting interests<sup>☆</sup>

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Received 27 August 2002; final version received 13 April 2004

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## Abstract

We analyze reputation effects in two-player repeated games of strictly conflicting interests. In such games, player 1 has a commitment action such that a best reply to it gives player 1 the highest individually rational payoff and player 2 the minmax payoff. Players have equal discount factors. With positive probability player 1 is a type who chooses the commitment action after every history. We show that player 1's payoff converges to the maximally feasible payoff when the discount factor converges to one. This contrasts with failures of reputation effects for equal discount factors that have been demonstrated in the literature.

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*JEL classification:* C71

*Keywords:* Reputation; Repeated games

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## 1. Introduction

We analyze reputation effects for a class of two-player repeated games with equal discounting. Our theorem provides a tight bound on the Nash equilibrium payoffs when the discount factor converges to one.

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<sup>☆</sup> Pesendorfer gratefully acknowledges the financial support from the NSF (under Grant SES-0236882). Dekel gratefully acknowledges financial support from the NSF (under Grant SES-0111830) and the Hammer Fund.

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In the seminal work of Fudenberg and Levine [7,8], one long-run player faces a sequence of short-run players who play the game only once. Fudenberg and Levine analyze a perturbed game of incomplete information in which there is positive prior probability that the long-run player is a “commitment type” who always plays a particular “commitment action.” Fudenberg and Levine show that if the long-run agent is sufficiently patient then her payoff at any Nash equilibrium is bounded below by what she could get by publicly committing to the commitment action. Below, we refer to this bound on the equilibrium payoffs of the perturbed game as the “reputation result”.

In repeated games with two long-run players the reputation result holds only for particular classes of games. In the following, player 1 refers to the player whose type is private information and who seeks to establish a reputation. A (stage) game has *conflicting interests* (Schmidt [9]) if a best reply to the optimal commitment action of player 1 yields the minmax payoff for player 2. A game has *strictly conflicting interests* (Chan [3]) if a best reply to the commitment action of player 1 yields the best feasible and individually rational payoff for player 1 and the minmax payoff for player 2.

Schmidt [9] showed a reputation result for games of *conflicting interests* with two long-run players under the assumption that player 2 remains impatient (i.e., player 2’s discount factor stays bounded away from 1) when the discount factor of player 1 converges to one. For the case of equal discount factors, Chan [3] obtained a folk theorem in perfect equilibrium strategies for all perturbed repeated games except those where the commitment action is a dominant action in the stage game or those with *strictly conflicting interests*. The contribution of this paper is to show that a reputation result holds for repeated games of strictly conflicting interest with equal discount factors.

We know of only three other reputation results with equal discounting. Schmidt [10] showed a sequential-equilibrium reputation result for finitely repeated bargaining games. Chan [3] obtained a perfect-equilibrium reputation result for games where the commitment action is a dominant action in the stage game. The results of Schmidt [10] and Chan [3] hold for a wide range of discount factors including equal discounting. For repeated games with no discounting, Cripps and Thomas [4] derived a partial reputation result for Nash equilibria.<sup>1</sup>

The reputation results in games with asymmetric discounting are robust to the introduction of two-sided uncertainty, while ours is not: to obtain a one-sided reputation result with equal discounting it is necessary that we allow *only* one-sided uncertainty. That is, we replace the asymmetry in discount factors used by Schmidt [9] with one-sided asymmetric information. We conjecture that a repeated game with two-sided strictly conflicting interests, two-sided uncertainty and equal discounting will have a unique equilibrium—similar to Abreu and Gul [1]—in which a war-of-attrition is played prior to one player revealing herself to be

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<sup>1</sup> We say partial because player 1 does not necessarily obtain the payoff he would receive from committing to a strategy, but there is a lower bound on his payoffs that is in general greater than the minmax payoff of the folk theorem.

normal, and once this has occurred an equilibrium of the game of one-sided incomplete information, as characterized below, is played.<sup>2</sup>

## 2. The model

We begin by giving the notation that describes the stage game and the unperturbed repeated game. Then, we will describe our equilibrium concept and define games of strictly conflicting interests. There are two players, called “one” (she) and “two” (he). They move simultaneously. Player  $i$ ,  $i \in \{1, 2\}$ , chooses an action  $a_i$  from the finite set  $A_i$ . (We will let  $A_i$  denote the set of mixed stage-game strategies,  $\alpha_i$ , for player  $i$ .) Player  $i$ ’s payoff when the players use the actions  $(\alpha_1, \alpha_2)$  are denoted  $g_i(\alpha_1, \alpha_2)$ . We will use  $\Gamma$  to denote this stage game. The stage-game minmax payoff for player  $i$  is denoted  $\hat{g}_i$ , that is,  $\hat{g}_i := \min_{\alpha_{3-i}} \max_{\alpha_i} g_i(\alpha_1, \alpha_2)$ ; we normalize  $\hat{g}_2 = 0$ .

We denote the set of the feasible payoffs in  $\Gamma$  by  $F$ ; that is,  $F$  is the convex hull of  $\{(g_1(a_1, a_2), g_2(a_1, a_2)) \mid (a_1, a_2) \in A_1 \times A_2\}$ . We will use  $G$  to denote the set of feasible and individually rational payoffs;  $G := F \cap \{(g_1, g_2) \mid g_1 \geq \hat{g}_1, g_2 \geq \hat{g}_2\}$ . The largest feasible and individually rational payoff for player 1 is denoted  $\bar{g}_1$ ;  $\bar{g}_1 := \max\{g_1 \mid (g_1, g_2) \in G\}$ . Finally, we will let  $M$  be an upper bound on the magnitude of the players’ payoffs;  $M > |g_i(a_1, a_2)|$  for  $i = 1, 2$  and all  $a_1, a_2$ .

A game has *strictly conflicting interests* if player 1 can commit to an action which is the best for her and the worst for her opponent. More precisely, a game has strictly conflicting interests if player 1 has an action to which 2’s best replies yield the payoffs  $(\bar{g}_1, \hat{g}_2)$ —the maximum feasible and individually rational payoff to 1 and the minmax to 2. Let  $a_1^*$  denote such a (pure) action. Furthermore, a game of strictly conflicting interests satisfies  $g_2 = \hat{g}_2$  for all  $(\bar{g}_1, g_2) \in G$ . Note that this is a genericity assumption that is implied, for instance, by assuming the game comes from a generic extensive-form game.<sup>3</sup> We will use  $l$  to denote the minimum loss player 2 can sustain from not playing the best response to  $a_1^*$ . The chain-store game and the repeated-bargaining game (Schmidt [10] pp. 341–343) both have strictly conflicting interests.

If a game has strictly conflicting interests there is a linear upper bound on the feasible payoffs to player 2 that passes through the point  $(\bar{g}_1, \hat{g}_2)$ . That is, there exists a finite  $\rho \geq 0$  such that  $g_2 \leq \hat{g}_2 + \rho(\bar{g}_1 - g_1)$ ,  $\forall (g_1, g_2) \in F$ . Given our normalization,  $\hat{g}_2 = 0$ , this reduces to

$$g_2 \leq \rho(\bar{g}_1 - g_1). \tag{1}$$

<sup>2</sup> Abreu and Gul consider a bargaining game in which the one-sided asymmetric information game has a unique solution, and show that the two-sided game has a unique war-of-attrition-like equilibrium. Abreu and Pearce [2] derive a similar result for general repeated games where they *assume* that a unique equilibrium must be played after all histories in which both players are revealed not to be the commitment type, which they show implies a unique equilibrium in the one-sided asymmetric information case. Using this they argue that the only robust equilibrium have a Nash-bargaining-with-endogenous-threats payoff. Our result essentially derives uniqueness in the one-sided case, and so suggests that for games with two-sided strictly conflicting interest Abreu and Pearce’s assumption of a unique equilibrium in the repeated symmetric-information game is not needed.

<sup>3</sup> We do not assume that the best response is unique.

The stage game above is played in each of the periods  $t = 0, 1, 2, \dots$ . The players have perfect recall and can observe the past pure actions chosen by their opponents. Let  $H^t := (A_1 \times A_2)^t$  denote the set of all partial histories,  $h^t$ , that can be observed by players before the start of period  $t$ .<sup>4</sup> A behavior strategy for player  $i$  in the game is a map  $\sigma_i : \cup_{t=0}^\infty H^t \rightarrow \mathcal{A}_i$ . A history  $h^\infty \in H^\infty$  will occasionally be denoted as a partial history  $h^t$  and its continuation  $h^{-t}$ , i.e.,  $h^\infty = (h^t, h^{-t})$ . The players' continuation payoffs in the repeated game given the partial history  $h^t$  are given by the normalized discounted sum of the continuation stage-game payoffs

$$g_i(h^\infty, t) := (1 - \delta) \sum_{s=t}^\infty \delta^{s-t} g_i(a_1^s, a_2^s),$$

where  $\delta < 1$  is the players' common discount factor. We will use  $\Gamma(\delta)$  to denote the discounted repeated game of complete information.

Now we will perturb the game  $\Gamma(\delta)$ . We will suppose that player 1 may be one of many different types. One of these is the “normal” type with the payoffs and actions described above, and a second is a “commitment” type that always plays the stage-game action  $a_1^*$ .<sup>5</sup> We denote this strategy in the repeated game by  $\sigma_1^*$ . We will not be precise about the remaining types—they may have different stage-game payoffs or just play given repeated-game strategies. The type of player 1 is chosen at time  $t = -1$  by Nature; with probability  $\mu$  Nature selects the commitment type, with probability  $1 - \mu - \phi$  Nature selects the normal type and with probability  $\phi$  Nature selects another type (possibly according to a distribution over other types). Player 1 observes the outcome of Nature's choice but player 2 does not. We will study the Nash equilibria of this repeated game of incomplete information. To do this it is convenient to specify player 2's priors after certain histories. Let  $\mu(h^t)$  denote player 2's prior that player 1 is a commitment type at the start of period  $t$  when the partial history of play is  $h^t \in H^t$  (Given  $\mu > 0$  and a repeated-game strategy of the normal type  $\mu(h^t)$  can be determined from Bayes' rule for any history in which player 1 has always used the action  $a_1^*$ .)

### 3. The result

This section begins with some intuitions and then presents our result. We show that if there is positive probability of the commitment type,  $\mu > 0$ , then player 1's equilibrium payoff is bounded below by a factor (depending on  $\mu$  and  $\phi$ ) which tends to  $\bar{g}_1$  as the players' common discount factor tends to unity and  $\phi$  tends to zero. Thus, as both players become very patient, at every equilibrium player 1 gets arbitrarily close to what she could receive if

<sup>4</sup> Define  $H^0$  to be an arbitrary singleton set.

<sup>5</sup> For our purposes, it will not matter whether the commitment type is an automaton who is “programmed” to play  $a_1^*$ , or comes with stage-game preferences that give her a payoff (independent of player 2's actions) from  $a_1^*$  strictly greater than her payoff from all other actions. The folk-theorem result of Cripps and Thomas [6] does not apply here because the existence of feasible and *strictly* individually rational payoffs for such a commitment type is violated. This is true, however, of all types with conflicting interests (not just those with strict conflicting interests).

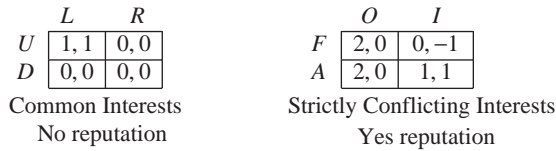


Fig. 1.

she publicly committed to playing  $a_1^*$  forever when there is a small amount of incomplete information.

For general stage games the result we seek may fail. The reason is that it can take the informed player so long to acquire a reputation that the costs of deviating from the equilibrium and thereby acquiring a reputation are not worth the benefits. Acquiring a reputation is costly in such equilibria because there are many periods in which the uninformed player does not play a best response to the commitment action. He does this because he is patient, believes he is most likely not facing the commitment type, and believes the non-commitment opponent is playing an equilibrium strategy that will ultimately reward him for the short-run costs of not playing a best response (or, equivalently, punish him for playing a best response). Provided these rewards (or punishments) are large enough and occur with a sufficiently high probability, the uninformed player is willing to not play a short-run best response for a very long time.

We show that reputations can be built in repeated games with strictly conflicting interests. The reason is that the above rewards must be given with a very high probability in such games. Hence, player 2 is never willing to play more than a finite number (which only depends on  $\mu$ ,  $\phi$  and the stage-game payoffs) of non-best responses to the commitment action. As the players become very patient those periods become an insignificant component of player 1’s payoffs, so her payoff from mimicking the commitment type at any equilibrium approaches her full reputation payoff.

It might appear counter-intuitive that reputation results obtain when the uninformed player receives the worst possible payoff in the reputation equilibrium. To gain some intuition for the role of the conflicting-interest assumption consider the following games (Fig. 1); Cripps and Thomas [5] have shown that there are many (perfect) equilibrium payoffs in the common-interest game on the left, while our result implies that in the chain-store game of strictly conflicting interests on the right player 1 can obtain a reputation and receives payoffs arbitrarily close to 2.

We highlight below one difference in the calculations involved in verifying whether a reputation effect can exist in these examples. We consider a particular type of equilibrium and a particular deviation. The condition that the deviation is not profitable in the common-interest game appears consistent with many initial periods in which 2 plays *R* and 1 plays *U* with a high probability. We show how this condition changes in the chain-store game in a way that imposes more restrictions on how often 1 can play *A* (and hence how often 2 can play *I*). The calculations are only suggestive of the difference; the proof that reputation effects do not survive in the common-interest game is contained in Cripps and Thomas [5], and that there is a reputation effect in the chain-store paradox follows from our proof below.

For the common-interest game on the left, Cripps and Thomas [5] construct equilibria in which player 1 mixes and player 2 chooses  $R$  for in the first  $T$  periods along a history where  $(U, L)$  is realized in every period. As the common discount factor converges to one,  $T$  converges to infinity sufficiently fast so that the payoff of player 1 stays bounded away from 1. Note that player 2's belief that 1 is the commitment type increases each time this mixed strategy is played and the realization is  $U$ . Therefore, a large  $T$  requires that player 1 chooses  $U$  with probability close to one while maintaining the incentive for player 2 to choose  $R$ . To see why this can be done in the common-interest game consider a belief for 2 such that if one more realization of  $U$  occurred, then 2 would play  $L$  in every subsequent period. For player 1 to mix and hence to be indifferent this implies that the payoffs to  $(U, R)$  and  $(D, R)$  are identical and equal to  $\delta(1, 1)$ . With what probability does player 1 have to play  $D$  to give player 2 an incentive to play  $R$ ? If  $(U, L)$  is played the continuation payoff is also  $(1, 1)$ , but after  $(D, L)$  player 2 can be punished by playing  $(D, R)$  in every period from then on which implies a payoff of 0. Thus there is a difference in player 2's long-run payoffs if he deviates from  $R$  in this period. Hence, it is enough that player 1 plays  $D$  with a probability proportional to  $1 - \delta$  to deter a deviation. Building on this observation Cripps and Thomas [5] show that the reputation result fails in the common-interest game.

Let us contrast this with the game of (strictly) conflicting interests (the chain-store game) on the right of Fig. 1. Again consider an equilibrium where player 1 is using a mixed strategy to give player 2 the incentives not to play the best response  $O(ut)$  but  $I(n)$  for one last period and a belief for 2 such that one more realization of  $F$  would lead player 2 to play  $O$  forever. When player 2 plays  $I(n)$  and player 1 randomizes she is again indifferent so her total payoffs to  $(F, I)$  and  $(A, I)$  are  $2\delta$ . The fact that player 1 must receive  $2\delta$  and the assumption of strictly conflicting interests forces player 2's total payoff to  $(A, I)$  to be very close to zero (less than  $\frac{1}{2}(1 - \delta)$ ). (To see this, note that we have  $(1 - \delta)(1, 1) + \delta(x, y) = (2\delta, z)$ , so  $x = \frac{2\delta - 1 + \delta}{\delta} = 2 - \frac{1 - \delta}{\delta}$ , so  $z \leq 1 - \delta + \frac{1 - \delta}{\delta} = \frac{1 - \delta^2}{\delta}$ ). The continuation payoffs player 2 receives from  $(F, O)$  and  $(A, O)$  are also at least zero because this is his minmax payoff. Player 2's long-term payoffs cannot, therefore, be used to provide an incentive for player 2 to play  $I$ . The incentive to play  $I$  can only result from the probability with which player 1 plays  $A$  and this is independent of  $\delta$ . We will show that a similar argument applies to earlier randomizations and that they too are independent of  $\delta$ . Hence, the number of periods in which player 1 can randomize remains fixed. In contrast to the preceding example, this yields a constraint on how many initial periods player 1 can randomize while player 2 plays  $I$ , and in turn this is what enables the reputation effect.

Our main result is that if the uncertainty about all types is sufficiently small and there is strictly positive prior probability of the commitment type, then the normal type's payoff at any equilibrium becomes arbitrarily close to his full reputation payoff as both of the players become patient. The rate of convergence here only depends on the parameters of the stage game. The proof is long and has been divided into several steps. We consider a pure strategy for player 2 that fails to play a best response to the commitment action in the most periods among all pure strategies that have positive probability in the equilibrium mixed strategy. Given the commitment action has been played until  $t - 1$ , player 2 must expect to receive at least 0 (his minmax) from future play of this strategy. This future payoff is made up

of continued play of the commitment action and, with some probability in period  $s \geq t$ , a deviation from the commitment action. At this point he only faces the normal type or the other types. Against the other types he can get at most  $M$  from period  $s$  on, however, in games of strictly conflicting interests there is a much tighter bound on what he receives from the normal types from  $s$  on. This is because the normal type must be indifferent between playing the commitment action for, say,  $n$  future periods in which player 2 does not play a best response and deviating from the commitment action. (This is a generalization of the  $\frac{1-\delta^2}{\delta}$  bound in the above example.) Writing the individual rationality condition for each such  $t$  gives a family of linear inequalities in the probabilities of deviations from the commitment action. In the first lemma we show that we only need pay attention to a finite number of such inequalities (for a given discount factor). The second lemma uses duality theory to write down a sufficient condition for the family of inequalities to have no solution. The final lemma shows this sufficient condition will always apply when there are many periods in which player 2 does not play a best response to the commitment type.

The reason our result holds only for small amounts of uncertainty about other types is because it may be that there are types present who will provide an incentive for player 2 to fight player 1's attempt to build a reputation. If such types are very likely, they make the costs to acquiring a reputation significant. For example, consider the case where player 2 attaches high probability to a type that rewards him greatly for not playing a best response to the commitment type in the first  $K = -\ln 2 / \ln \delta$  periods. Player 2 would attach high probability to receiving a reward discounted by  $\frac{1}{2} = \delta^K$ , and this would give him a non-vanishing incentive to make player 1 wait at least  $K$  periods to gain a full reputation. Thus, the normal type could expect at most  $\bar{g}_1 - \frac{1}{2}c$  where  $c$  was the payoff cost from player 2's actions in the early periods of play. However, this argument relies on player 2 being convinced that it is very likely that there are such reward types present. If the amount of the overall incomplete information is small, i.e., there is a small perturbation of the incomplete information game, this cannot be the case.

To state our result, given a game of strictly conflicting interests  $\Gamma$  let  $f : \mu \mapsto \exp\left(\frac{4(l+2M\rho)}{l\mu}\right)$  and  $b : \mu \mapsto \frac{Mf(\mu)}{l(1-\mu)}$ . (Recall that  $M, \rho$  and  $l$  are parameters determined by  $\Gamma$ ; they are defined in the second and third paragraphs of Section 2.)

**Proposition 1.** *Let  $\Gamma$ , a game of strictly conflicting interests, be given. Then the normal type of player 1's payoff at any Nash equilibrium of the repeated of incomplete information is bounded below by*

$$\bar{g}_1 - 2M \left(1 - \delta^{f(\mu)} e^{-\frac{\phi}{\delta} b(\mu)}\right).$$

Before proving our result (which will take the rest of the paper) we will state a trivial two-part corollary to the proposition. (1) With only two types, as both players become very patient the normal type must get the full-reputation payoff at any equilibrium. (2) Given any prior  $\mu$  on the commitment type the normal type's payoff at any equilibrium can be made arbitrarily close to her full-reputation payoff as the players become patient and the probability of other types is made small.

**Corollary 1.** *Let  $\Gamma$ , a game of strictly conflicting interests,  $\mu > 0$ , a probability of the commitment type, and  $\varepsilon > 0$ , be given: (1) If  $\phi = 0$ , then the normal type's payoff at any Nash equilibrium approaches  $\bar{g}_1$  as  $\delta \rightarrow 1$ . (2) There exists  $\underline{\delta} < 1$  and  $\underline{\phi} > 0$  such that at any Nash equilibrium of the game with  $\delta > \underline{\delta}$  and  $\phi < \underline{\phi}$  the normal type's payoff is bounded below by  $\bar{g}_1 - \varepsilon$ .*

**Proof of Proposition 1.** Let  $\mu > 0$ ,  $\delta < 1$ ,  $\phi$  and any equilibrium of the repeated game be given. The proof of this result will proceed in several steps. (Steps 1–2) Using the fact that player 2's payoffs in all subgames are at least the minmax we find a family of inequalities, (4), that the normal type's mixed strategy will satisfy in any equilibrium. (Step 3) We then combine this with the linear bound (1), to obtain bounds based on player 1's payoffs that yield inequalities (5), that the normal type's mixed strategy satisfy in equilibrium. (Step 4) We refine these inequalities by arguing in Lemma 1 that when player 1 plays  $a_1^*$  repeatedly then on all equilibrium paths there is a finite last period after which payoffs are  $(\bar{g}_1, 0)$  in every period, and in the period preceding the maximal such period player 2 does not play a best reply. (Step 5) We then restate the inequalities in Lemma 2 using Farkas' lemma to find inequalities that must be violated in any equilibrium. (Step 6) Finally, in Lemma 3 we show that if player 2 fails to play a best reply to the commitment strategy too often, there is a solution to the inequalities of Lemma 2, implying that this cannot happen in equilibrium.

*Step 1:* Player 2's equilibrium payoffs are at least the minmax payoff.

Consider a history in which player 1 plays  $a_1^*$  in all periods prior to period  $t$ . To find the main inequalities, (4), we give an upper bound on player 2's equilibrium payoffs from period  $t$  onward. This upper bound can be decomposed into the per-period payoffs obtained so long as player 1 plays  $a_1^*$  and the continuation payoffs once 1 departs from the commitment strategy and plays anything else. Let  $g_{2\tau}^e$  denote the per-period payoffs against  $a_1^*$  and note that  $g_{2\tau}^e \leq 0$  since  $g_{2\tau}^e$  is bounded above by the minmax payoff. Let  $c_\tau$  denote the continuation payoff of player 2 conditional on facing the normal type starting from a first departure by player 1 from the commitment strategy in period  $\tau$  ( $c_\tau$  includes the payoffs to 2 in period  $\tau$ ). Player 2's continuation payoff from the other potential types may depend on the type faced but are bounded by  $M$ . Let  $\sigma_2$  denote a pure strategy in the support of player 2's equilibrium strategies. Let  $\pi_\tau$  denote the probability of facing the normal type and the normal type playing the commitment action up to but not including period  $\tau$  along the path generated by  $\sigma_2$  and the equilibrium strategy of player 1. Let  $\xi_\tau$  denote the probability of facing any other type with the property that the commitment action is played up to but not including period  $\tau$  along the path generated by  $\sigma_2$  and the equilibrium strategy of player 1. The total payoff must be greater than 2's minmax payoff (of zero), which gives us the following inequalities for all  $t < T$ , where  $T$  (which at this point may be  $\infty$ ) denotes the first period after which the payoffs to the normal type are  $\bar{g}_1$  along the path generated by  $(\sigma_1^*, \sigma_2)$ .<sup>6</sup> Thus after  $T$  periods of observing the commitment action player 2 receives the

<sup>6</sup> We adopt the convention  $\sum_{r=t}^{t-1} \delta^r = 0$ .



payoff zero if he faces the commitment type and at most  $M$  from the other types.

$$\begin{aligned}
 0 \leq & \sum_{s=t}^{T-1} \pi_s \left[ (1 - \delta) \sum_{r=t}^{s-1} \delta^{r-t} (g_{2r}^e) + \delta^{s-t} c_s \right] \\
 & + \sum_{s=t}^{T-1} \xi_s \left[ (1 - \delta) \sum_{r=t}^{s-1} \delta^{r-t} (g_{2r}^e) + \delta^{s-t} M \right] \\
 & + \left( 1 - \sum_{s=0}^{T-1} (\pi_s + \xi_s) \right) (1 - \delta) \sum_{r=t}^{T-1} \delta^{r-t} (g_{2r}^e) + \delta^{T-t} M \left( \phi - \sum_{s=0}^{T-1} \xi_s \right). \quad (2)
 \end{aligned}$$

Along this path after time  $T$  player 2 can get at most zero against the normal or commitment type, but may be able to get at most  $M$  against the other types. The above assumes player 2 receives the upper bound on payoffs when any other type is present after time  $T$ .

*Step 2:* Simplifying the resulting inequalities (2); some algebra.

Now, we will find a necessary condition for there to be a solution to the inequalities (2). The sum of the coefficients of  $\xi_{T-1}$  in (2) are positive (i.e., the RHS increases in  $\xi_{T-1}$ ) and there is a constraint  $\sum_{s=0}^{T-1} \xi_s \leq \phi$ . Thus given any solution  $(\pi_s$  and  $\xi_s)$  to (2) there is another solution with  $\sum_{s=0}^{T-1} \xi_s = \phi$  and no fourth term. Moreover the third term is not positive ( $g_{2r}^e \leq 0$ ) and  $\sum_{t=0}^{T-1} (\pi_t + \xi_t) \leq 1 - \mu$  so, after dividing by  $(1 - \delta)\delta^{-t}$ , a necessary condition for the existence of a solution to the above inequalities is for there to exist a solution to

$$\begin{aligned}
 -\mu \sum_{r=t}^{T-1} \delta^r (g_{2r}^e) \leq & \sum_{s=t}^{T-1} \pi_s \left[ \sum_{r=t}^{s-1} \delta^r (g_{2r}^e) + \frac{\delta^s c_s}{1 - \delta} \right] \\
 & + \sum_{s=t}^{T-1} \xi_s \left[ \sum_{r=t}^{s-1} \delta^r (g_{2r}^e) + \frac{\delta^s M}{1 - \delta} \right] \quad \forall t < T. \quad (3)
 \end{aligned}$$

The LHS of (3) decreases in  $g_{2r}^e$  and the RHS increases in  $g_{2r}^e$ . Thus replacing  $g_{2r}^e$  in (3) with a larger number cannot violate the inequalities (3). Let  $\iota_r$  be an indicator function, such that  $\iota_r = 1$  iff a short-run best reply is not played ( $g_{2r}^e < 0$ ) and  $\iota_r = 0$  iff a best response is played ( $g_{2r}^e = 0$ ). Let  $-l < 0$  be the upper bound on the non-zero values of  $g_{2r}^e$  (as discussed in the paragraph preceding (1) such a bound exists as the stage game is finite), so  $g_{2s}^e \leq -\iota_s l$  for all  $s$ . Finally, let  $w_t \equiv \sum_{r=t}^{T-1} \delta^r \iota_r$  so that  $-lw_t \geq \sum_{r=t}^{T-1} \delta^r g_{2r}^e$  and  $-l(w_t - w_s) \geq \sum_{r=t}^{s-1} \delta^r g_{2r}^e$ . Thus, if  $k \equiv M/l$ , a necessary condition for (3) is

$$\mu w_t \leq \sum_{s=t}^{T-1} \pi_s \left[ w_s - w_t + \frac{\delta^s c_s}{l(1 - \delta)} \right] + \sum_{s=t}^{T-1} \xi_s \left[ w_s - w_t + \frac{\delta^s k}{1 - \delta} \right] \quad \forall t < T. \quad (4)$$

*Step 3:* Using (4) and (1) to obtain inequalities on player 1’s payoff by considering paths with the maximal number of periods in which player 2 does not play a best reply to  $a_1^*$ .

The above applies to any pure strategy in the support of player 2’s equilibrium strategy. Now we find a lower bound on player 1’s payoffs in any period in which  $\pi_s > 0$ . To do

this we choose a pure strategy  $\sigma'_2$  of player 2 in the support of 2's equilibrium strategy that has the most periods in which 2 does not play a best reply to 1's commitment action. Let  $n(t)$  denote the number of times player 2 does not play a best response to  $a_1^*$  until period  $t$  (when 2 plays  $\sigma'_2$  and 1 plays  $a_1^*$  in each period). If the history generated by these two strategies occurred until period  $s$ , then player 1 expects there to be at most  $n(T) - n(s)$  periods in the future in which player 2 does not play a best response to  $a_1^*$ . (If it were possible for player 2 to not play a best response more than  $n(T) - n(s)$  periods, then there must exist a strategy  $\sigma''_2$  played in equilibrium that agrees with  $\sigma'_2$  until  $s$  and has more periods of not playing a best response to the commitment type, contradicting the definition of  $\sigma'_2$ .) By continuing to play  $a_1^*$  forever after player 1 can expect to get a payoff of at least  $(1 - \delta c^{n(T)-n(s)})(-M) + \delta^{n(T)-n(s)} \bar{g}_1$ . (This assumes the  $n(T) - n(s)$  periods of not playing a best response occur immediately (the worst possible case) and that the loss from this is maximized.) This lower bound is weakly greater than  $\bar{g}_1 - [1 - \delta^{n(T)-n(s)}]2M$ . When  $\pi_s > 0$ , this is also a lower bound on what player 1 can expect to get from her equilibrium action of not playing  $a_1^*$  after the  $s$  period history generated by  $(\sigma_1^*, \sigma'_2)$ .

The profile of normal-type and player 2 payoffs when the normal type of player 1 does not play  $a_1^*$  must lie in the feasible set  $F$ . Eq. (1) and the lower bound on the normal type's payoffs from not playing  $a_1^*$  therefore imply an upper bound on player 2's payoff  $c_s$  whenever  $\pi_s > 0$ :  $c_s \leq 2M\rho [1 - \delta^{n(T)-n(s)}]$ . (Recall the normalization  $\hat{g}_2 = 0$ .) If these upper bounds are included in the inequalities (4), we get the following necessary condition for all  $t < T$ :

$$\begin{aligned} \mu w_t \leq & \sum_{s=t}^{T-1} \pi_s \left[ w_s - w_t + h\delta^s \left( \frac{1 - \delta^{n(T)-n(s)}}{1 - \delta} \right) \right] \\ & + \sum_{s=t}^{T-1} \xi_s \left[ w_s - w_t + \frac{k\delta^s}{1 - \delta} \right] \quad \forall t < T. \end{aligned} \tag{5}$$

Here  $h \equiv 2M\rho/l$  and we can include the upper bound on  $c_s$  even for periods when  $\pi_s = 0$  because (4) is independent of  $c_s$  in such periods.

*Step 4:* For any given  $\delta$  and equilibrium there is a finite period,  $T$ , (as defined above) after which payoffs are  $\bar{g}_1$ ,<sup>7</sup> and  $\sigma'_2$  does not play a best response to the commitment action in period  $T - 1$ .

**Lemma 1.** For a given  $\delta < 1$  and  $\sigma'_2$ , (i)  $T$  is finite, and (ii)  $v_{T-1} = 1$ .

**Proof of (i).** Suppose not, and  $T$  is infinite or (5) holds for all  $t \geq 0$ . As  $t \rightarrow \infty$  the sums  $\sum_{s=t}^{T-1} \pi_s$ ,  $\sum_{s=t}^{T-1} \xi_s$  converge to zero, because  $\sum_{s=0}^{T-1} \xi_s$ ,  $\sum_{s=0}^{T-1} \pi_s \leq 1$ . Further, the first term in braces on the RHS of (5) is bounded above by  $\delta^t h / (1 - \delta)$  and the second by  $\delta^t k / (1 - \delta)$ , so as  $t \rightarrow \infty$  the RHS of (5) is strictly less than  $\mu\delta^t$ . The LHS is bounded

<sup>7</sup>This says that eventually play will involve the commitment action forever; this feature is stronger than, say, the Fudenberg and Levine [7] reputation result because of the special structure of our game.

below by  $\mu_t \delta^t$ . If  $T$  is infinite there are infinitely many  $t$ 's such that  $l_t = 1$ , and there continue to be values  $t$  for which the LHS of (4) is greater than  $\mu \delta^t$ , which yields a contradiction.  $\square$

**Proof of (ii).** Suppose that  $l_{T-1} = 0$ . If  $\sigma'_2$  and the commitment action has been played until  $T - 1$ , then in period  $T - 1$  player 2's equilibrium behavior strategy after this partial history is a best reply. (If he did not best reply with probability one, there would exist a pure strategy for player 2 that plays more non-best responses than  $\sigma'_2$ —play  $\sigma'_2$  and do not play a best response in period  $T - 1$ .) If player 2 plays a best response with probability one, then the normal type's continuation payoff at the start of period  $T - 1$  (after this partial history) is also  $\bar{g}_1$ . By playing  $a'_1$  after the partial history player 1 gets  $\bar{g}_1$  in the stage game (because  $l_{T-1} = 0$  and all best replies give her the commitment payoff) and as a continuation payoff (by definition of  $T$ ). Any equilibrium action,  $a'_1 \neq a_1^*$ , played with positive probability after the partial history will also give her the payoff  $\bar{g}_1$  (by indifference among actions played with positive probability in equilibrium). This is a contradiction because  $T$  is defined as the first time when the normal type has the continuation payoff  $\bar{g}_1$ .  $\square$

*Step 5:* Using the finiteness of  $T$  we use Farkas Lemma to determine when the system (5) of linear inequalities in the variables  $\pi \equiv (\pi_0, \dots, \pi_{T-1})'$  and  $\xi \equiv (\xi_0, \dots, \xi_{T-1})'$  has no solution.<sup>8</sup>

**Lemma 2.** *The inequalities (5) do not have a solution  $\pi \in \mathfrak{R}_+^T$ ,  $\xi \in \mathfrak{R}_+^T$  satisfying  $\sum_{s=0}^{T-1} \pi_s \leq 1 - \mu - \phi$  and  $\sum_{s=0}^{T-1} \xi_s = \phi$ , if there exists  $(x_0, \dots, x_{T-1}) \in \mathfrak{R}_+^T$  such that*

$$\frac{\mu}{2\delta^t \left[ A(n(T) - n(t)) + \frac{\phi k}{1-\delta} \right]} > \sum_{s=0}^t \frac{x_s}{w_s}, \quad t = 0, \dots, T - 1, \tag{6}$$

where  $\sum_{t=0}^{T-1} x_t = 1$  and  $A = (1 - \mu - \phi)(1 + h) + \phi$ .

**Proof.** See the appendix for the algebra.  $\square$

*Step 6:* The final step is a technical argument which shows that (6) must have a solution if  $n(T)$  is sufficiently large.

**Lemma 3.** *The inequalities (6) have a non-negative solution satisfying  $\sum_{t=0}^{T-1} x_t = 1$  if*

$$\delta^{n(T)} \leq \delta^f(\mu) e^{-\frac{\phi}{\delta} b(\mu)}. \tag{7}$$

**Proof.** See appendix for the algebra.  $\square$

*Step 7:* Combining the arguments.

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<sup>8</sup> We use primes to denote transposes.

To complete the proof of Proposition 1 notice that if (7) holds then by Lemmas 2 and 3 there can be no solution to (5). Thus there cannot exist an equilibrium where (7) holds. This implies a lower bound on  $\delta^{n(T)}$  at every equilibrium. As the normal type can ensure a payoff of at least  $(1 - \delta^{n(T)})(-M) + \delta^{n(T)}\bar{g}_1$  at every equilibrium this implies the normal type's payoff at every equilibrium is bounded below by

$$\bar{g}_1 - 2M \left(1 - \delta^{f(\mu)} e^{-\frac{\phi}{\delta} b(\mu)}\right). \quad \square$$

**Appendix**

**Proof of Lemma 2.** Eq. (5) can be written as the matrix inequality  $b \leq X\pi + Y\xi$  where  $X$  is an upper-triangular matrix with  $\frac{h}{1-\delta}\delta^s \left(1 - \delta^{n(T)-n(s)}\right) + w_s - w_t$  in the  $(t + 1, s + 1)$ th entry,  $Y$  is an upper-triangular matrix with  $\frac{k}{1-\delta}\delta^s + w_s - w_t$  in the  $(t + 1, s + 1)$ th entry, and  $b = (\mu w_t)_{t=0}^{T-1}$ . (Notice that the indices  $s$  and  $t$  run from zero while as usual the rows and columns of matrices run from one.) If (5) has a non-negative solution, then the equations  $b = X\pi + Y\xi - \omega$  have a solution  $(\pi'; \xi'; \omega') \in \mathfrak{R}_+^{3T}$ . The constraints  $e'_T \pi \leq 1 - \mu - \phi$  and  $e'_T \xi = \phi$ , where  $e'_T = (1, 1, \dots, 1) \in \mathfrak{R}^T$ , can be written as  $e'_T \pi + z = 1 - \mu - \phi$  and  $e'_T \xi = \phi$  for some  $z \geq 0$ . Therefore, (5) has a solution satisfying the conditions in Lemma 2 if and only if (8) below has a solution.

$$\begin{pmatrix} b \\ 1 - \mu - \phi \\ \phi \end{pmatrix} = \begin{bmatrix} X & Y & -I_T & \mathbf{0} \\ e'_T & \mathbf{0}' & \mathbf{0}' & 1 \\ \mathbf{0}' & e'_T & \mathbf{0}' & 0 \end{bmatrix} \begin{pmatrix} \pi \\ \xi \\ w \\ z \end{pmatrix}, \quad \begin{pmatrix} \pi \\ \xi \\ w \\ z \end{pmatrix} \in \mathfrak{R}_+^{3T+1}. \quad (8)$$

(Here  $I_T$  denotes the  $T$  dimensional identity matrix, and  $\mathbf{0}$  a column vector of zeroes.)  $T$  is finite so, by Farkas' Lemma, there is no solution to (8) if and only if there exists  $y' := (y_0, y_1, \dots, y_{T+1}) \in \mathfrak{R}^{T+2}$  such that

$$y' \begin{pmatrix} b \\ 1 - \mu - \phi \\ \phi \end{pmatrix} > 0, \quad y' \begin{bmatrix} X & Y & -I_T & \mathbf{0} \\ e'_T & \mathbf{0}' & \mathbf{0}' & 1 \\ \mathbf{0}' & e'_T & \mathbf{0}' & 0 \end{bmatrix} \leq \mathbf{0}. \quad (9)$$

The first inequality in (9) gives (10) below and the second decomposes into (11), (12) and  $(y_0, \dots, y_{T-1}) \geq \mathbf{0}$  and  $y_T \leq 0$ .

$$\mu \sum_{s=0}^{T-1} y_s w_s > -(1 - \mu - \phi)y_T - \phi y_{T+1}, \quad (10)$$

$$-y_T \geq \sum_{s=0}^t y_s \left[ h\delta^t \left( \frac{1 - \delta^{n(T)-n(t)}}{1 - \delta} \right) + w_t - w_s \right], \quad t = 0, \dots, T - 1, \quad (11)$$

$$-y_{T+1} \geq \sum_{s=0}^t y_s \left[ \frac{k\delta^t}{1 - \delta} + w_t - w_s \right], \quad t = 0, \dots, T - 1. \quad (12)$$

Let  $\hat{y}^* \equiv (y_0^*, \dots, y_{T-1}^*) \in \mathfrak{N}_+^T$  denote a solution to (13) for  $t = 0, \dots, T - 1$ .

$$\begin{aligned} \frac{\mu}{2} \sum_{s=0}^{T-1} y_s w_s &> (1 - \mu - \phi) \sum_{s=0}^t y_s \left[ h\delta^t \left( \frac{1 - \delta^{n(T)-n(t)}}{1 - \delta} \right) + w_t \right] \\ &+ \phi \sum_{s=0}^t y_s \left[ \frac{k\delta^t}{1 - \delta} + w_t \right]. \end{aligned} \tag{13}$$

We aim to show that  $y^* = (\hat{y}^*, y_T^*, y_{T+1}^*)$  for suitably chosen  $y_T^*, y_{T+1}^*$  is a solution to (9), so that (8) or (5) have no solution. Choose  $-y_{T+1}^*$  to equal the maximum (over  $t$ ) of the expressions on the RHS of (12) evaluated at  $y^*$ . As  $\hat{y}^*$  satisfies (13) the maximum (over  $t$ ) of the RHS of (13) (evaluated at  $\hat{y}^*$ ) is less than  $\frac{1}{2}\mu \sum_{s=0}^{T-1} y_s^* w_s$ . Hence,  $\phi$  times the maximum of the RHS of (12) is also less than  $\frac{1}{2}\mu \sum_{s=0}^{T-1} y_s^* w_s$ , that is,  $\frac{1}{2}\mu \sum_{s=0}^{T-1} y_s^* w_s > \phi(-y_T^*)$ . Similarly, choosing  $-y_T^*$  to equal zero or the maximum (over  $t$ ) of the expressions on the RHS of (11) evaluated at  $\hat{y}^*$  implies that  $\frac{1}{2}\mu \sum_{s=0}^{T-1} y_s^* w_s > (1 - \mu - \phi)(-y_{T+1}^*)$ . Combining these two gives  $\mu \sum_{s=0}^{T-1} y_s^* w_s > -\phi y_T^* - (1 - \mu - \phi)y_{T+1}^*$ . Thus, if (13) has a solution  $\hat{y}^* \geq 0$  there is also a solution to (9).

As  $(y_0^*, \dots, y_{T-1}^*) \neq \mathbf{0}$  at any solution to (13) and  $w_t > 0$  (as  $t_{T-1} = 1$  from Lemma 1) we can define  $x_t \equiv y_t^* w_t / (\sum_{s=0}^{T-1} y_s^* w_s)$  which has the property that  $\sum_{t=0}^{T-1} x_t = 1$ . Dividing (13) now gives

$$\begin{aligned} \frac{\mu}{2} &> (1 - \mu - \phi) \sum_{s=0}^t \frac{x_s}{w_s} \left[ h\delta^t \left( \frac{1 - \delta^{n(T)-n(t)}}{1 - \delta} \right) + w_t \right] \\ &+ \phi \sum_{s=0}^t \frac{x_s}{w_s} \left[ \frac{k\delta^t}{1 - \delta} + w_t \right], \quad t < T. \end{aligned} \tag{14}$$

As  $x_s \geq 0$  and  $w_t = \sum_{r=t}^{T-1} \delta^r t_r \leq \delta^t (1 - \delta^{n(T)-n(t)}) / (1 - \delta) \leq \delta^t (n(T) - n(t))$  (this puts the  $n(T) - n(t)$  times when  $t_r = 1$  as early as possible), we can substitute these upper bounds and derive a sufficient condition for (14). This is (6).  $\square$

**Proof of Lemma 3.** Define  $(z_0, z_1, \dots, z_{T-1}) \geq 0$  to be the solution to the equations

$$\frac{\mu}{4\delta^t \left[ A(n(T) - n(t)) + \frac{\phi k}{1 - \delta} \right]} = \sum_{s=0}^t z_s, \quad t = 0, \dots, T - 1. \tag{15}$$

This solution is non-negative because the ratio on the LHS of (15) is increasing in  $t$  and for  $t = 0$  the LHS of (15) is positive. Notice that  $x_s^* \equiv z_s w_s$  (for  $s = 0, \dots, T - 1$ ) is a

solution to the inequalities (6). However,

$$\sum_{t=0}^{T-1} x_t^* = \sum_{t=0}^{T-1} z_s \sum_{r=s}^{T-1} \delta^r l_r = \sum_{t=0}^{T-1} \delta^t l_t \sum_{s=0}^t z_s,$$

where the last equality holds by reversing the order of the summations. A substitution from (15) into the above implies

$$\begin{aligned} \sum_{t=0}^{T-1} x_t^* &= \sum_{t=0}^{T-1} \frac{\mu l_t}{4A[n(T) - n(t)] + \frac{4\phi k}{1-\delta}} \\ &= \sum_{n=1}^{n(T)} \frac{\mu}{4An + \frac{4\phi k}{1-\delta}} \\ &\geq \frac{\mu}{4A} \ln \left( 1 + \frac{An(T)}{A + \frac{\phi k}{(1-\delta)}} \right). \end{aligned}$$

The second line above follows from deleting elements when  $l_t = 0$ . The third line uses the fact that  $\ln m = \int_1^m dx/x \leq \sum_{x=1}^{m-1} 1/x$ . A sufficient condition for  $\sum_{t=0}^{T-1} x_t^* \geq 1$  is, therefore,  $An(T) \geq [A + \phi k/(1 - \delta)] \exp(4A/\mu)$ . However,  $1 - \delta \geq -\delta \ln \delta$ , so it is sufficient for  $n(T) \ln \delta \leq [\ln \delta - (\phi k/(A\delta))] \exp(4A/\mu)$ . Notice that  $1 - \mu \leq A \leq 1 + h$ , this implies a sufficient condition for (14) to have a solution that satisfies  $\sum_{t=0}^{T-1} x_t \geq 1$  is (7). However, the existence of a solution to (14) with  $\sum_{t=0}^{T-1} x_t \geq 1$  obviously implies the existence of a solution with  $\sum_{t=0}^{T-1} x_t = 1$ .  $\square$

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