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An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom*

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The independence axiom used to derive the expected utility representation of preferences over lotteries is replaced by requiring only convexity, in terms of probability mixtures, of indifference sets. Two axiomatic characterizations are proven, one for simple measures and the other continuous and for all probability measures. The representations are structurally similar to expected utility, and are unique up to a generalization of affine transformations. First-order stochastic dominance and risk aversion are discussed using a method which finds an expected utility approximation to these preferences without requiring differentiability of the preference functional. *Journal of Economic Literature* Classification Numbers: 022, 026.

1. INTRODUCTION

This paper provides an implicit representation for an axiomatic characterization of preferences under uncertainty. Essentially only the controversial independence axiom is changed to the substantially weaker betweenness axiom (Chew [2]), keeping ordering, monotonicity, and continuity type axioms. The betweenness axiom only requires that indifference sets be convex, i.e., if an individual is indifferent between two lotteries, then any probability mixture of these two is equally good. This characterization is of interest for a number of reasons. The betweenness axiom is appealing from a normative viewpoint but is compatible with behavior which is not permitted in expected utility, such as the Allais paradox. It also provides a useful behavioral approach since it is the weakest form under which preferences are both quasiconcave and quasiconvex. Quasiconcavity is

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necessary for the proof of existence of a Nash equilibrium, since if preferences are strictly quasiconvex anywhere then a mixed strategy is worse than one of the pure strategies used with positive probability in that mixed strategy. Furthermore, quasiconcavity together with risk aversion are sufficient conditions for continuity of asset demands (while risk aversion alone is not sufficient (Dekel [4])). Quasiconvexity on the other hand is necessary and sufficient for dynamic consistency of choices under uncertainty (see Green [10]). In general, when temporal decisions are made—given underlying expected utility preferences—the induced preferences will be quasiconvex (see Kreps and Porteus [11], and Machina [13]). Thus, in order to guarantee the existence of a Nash equilibrium, dynamic consistency, and continuous asset demands, we may want to impose quasiconcavity and quasiconvexity of preferences, giving betweenness—without making the additional restrictions necessary for expected utility.

The paper begins by presenting the axioms and characterization, discussing recent literature, and proving the representation. Two approaches are taken, one with a weak continuity axiom provides a representation for all simple probability measures (those whose support is a finite subset of the outcome set), and the second imposes a stronger form of continuity which suffices both to extend the results to the set of all distributions and also implies that the functional representation is continuous. Then an example is constructed to show that preferences may satisfy the axioms yet not have any differentiable preference functional, even when preferences are over the simplex (hence trivially continuous). This implies that the generalization of local monotonicity and risk aversion to global conclusions as proven in Machina [12] might not hold for all preferences of the type discussed here. However, an alternative and intuitive extension of local properties is demonstrated by examining the slopes of the indifference hyperplanes.

2. Axiomatic Characterization

There is an underlying compact metric space W which is the space of outcomes of lotteries, representing, for example, monetary outcomes or commodity bundles. Preferences, $\succsim$, are defined on the space of all probability measures (D) or simple probability measures (D_0) on the Borel field of W. Convex subsets of D and D_0 could also be dealt with; the details are not presented. From these preferences define the induced strict preference, $\succ$, and indifference, $\sim$, relations. Preferences over D and D_0 also induce preferences over W, where for any $w, w' \in W$, w is preferred to w' if the measure assigning probability 1 to w is preferred to the measure assigning probability 1 to w'. These measures will be denoted w, w' and this preference relation is written as $\succ$ where no confusion should result, and the context will clarify whether $w \in W$ (the outcome) or $w \in D$ (the degenerate measure) is implied. For any w, w' in W the measure which assigns probability α to w and $1-\alpha$ to w' is denoted $(\alpha, w; (1-\alpha), w')$.

The following axioms will be used (where P, Q, R are measures in D and $\bar{w}$, $\mathbf{w}$, w', and w'' are outcomes in W).

A1. (a) $\succ$ is a weak order ($\succsim$ is complete, $\succ$ is asymmetric, and both $\succ$ and $\sim$ are transitive).

(b) There exist best and worst elements in D_0 which are the sure outcomes denoted by $\bar{w}$ and $\mathbf{w}$. (These are not necessarily unique.)

A2. Solvability: If $P \succ Q \succ R$, then there exists an $\alpha \in (0, 1)$ such that $\alpha P + (1-\alpha) R \sim Q$.

A3. Monotonicity: If $w = \mathbf{w}$ or $w = \bar{w}$ and $w' \succ w''$ (resp. $w' \sim w''$), then $(\alpha, w'; (1-\alpha), w) \succ (\alpha, w''; (1-\alpha), w)$ for every $\alpha \in (0, 1)$ (resp. $(\alpha, w'; (1-\alpha), w) \sim (\alpha, w''; (1-\alpha, w)$ for every $\alpha \in [0, 1]$).

A4. Betweenness: If $P \succ Q$ (resp. $P \sim Q$), then $P \succ \alpha P + (1-\alpha) Q \succ Q$ for every $\alpha \in (0, 1)$ (resp. $P \sim \alpha P + (1-\alpha) Q$ for every $\alpha \in [0, 1]$).

PROPOSITION 1. *Preferences over* D_0 *satisfy* A1–A4 *if and only if there exists a function* $u(\cdot, \cdot)$: $W \times [0, 1] \to \mathbb{R}$ *increasing in the preference ordering on* W, *and continuous in the second argument such that* $P \succ Q$ (*resp.* $P \sim Q$) $\Leftrightarrow V[P] > V[Q]$ (*resp.* $V[P] = V[Q]$), *where* $V[F]$ *is defined implicitly as the unique* $v \in [0, 1]$ *that solves*

$$\int u(w, v)\, dF(w) = v u(\bar{w}, v) + (1-v)\, u(\mathbf{w}, v). \qquad (*)$$

Furthermore $u(w, v)$ *is unique up to positive affine transformations which are continuous functions of* v. *A particular transformation exists setting* $u(\mathbf{w}, v) = 0$ *and* $u(\bar{w}, v) = 1$ *for every* v, *givingthe simpler representation* (*similar to expected utility*)

$$\int u(w, V[F])\, dF(w) = V[F]. \qquad (**)$$

The uniqueness characterization of $u(\cdot, \cdot)$ in Proposition 1 is a natural extension of the result in expected utility theory that the Bernoulli utility function is unique up to affine transformations to the framework developed in this paper. To clarify this generalization let $V[F]$ be uniquely defined from $u(\cdot, \cdot)$ by (*) and let $\hat{V}[F]$ be uniquely defined by (*), where $\hat{u}(\cdot, \cdot)$ replaces $u(\cdot, \cdot)$. I say that $u(\cdot, \cdot)$ is unique up to positive affine transformations which are continuous functions of v when $V[F]$ and $\hat{V}[F]$ represent the same preferences if and only if $\hat{u}(w, v) = a(v)\, u(w, v) + b(v)$ for some $a(v)$ positive continuous and $b(v)$ continuous.

Monotonicity (A3) is a weaker axiom than the standard first-order stochastic dominance axioms (cf. [2, Property 3]). However, as will be seen in Section 4, A1–A4 are sufficient to prove that the preferences are first order stochastic dominance preserving. In the appendix I provide a characterization which does not assume A3. This characterization is similar to Proposition 1, except that $u(w, v)$ is not necessarily increasing in w (see Sect. 3.A).

The characterization in Proposition 1 is an implicit expected utility representation, and the similarity of equation (**) to an expected utility calculation suggests that results from the theory of expected utility can be extended to the framework of this paper. A general result along these lines, based on Epstein's observation [15] that many properties of an optimal choice depend on the indifference curve through that choice and not on the whole indifference map, can be derived. Let U be the set of real valued functions on W, $\bar{U}$ a subset of U and $\bar{D}$ a subset of D. Consider any proposition in expected utility theory of the following form: if $u(\cdot) \in \bar{U}$ then the distribution $F \in D$ which maximizes $\int u(w)\, dF(w)$ is in $\bar{D}$. For example, if u is concave then F is not second order stochastically dominated, and if u also has positive third derivative then the optimal F isn't third order stochastically dominated. This proposition can be extended to implicit expected utility preferences as follows: if $u(\cdot, v) \in \bar{U}$ for every v then the F which maximizes (**) is in $\bar{D}$. This claim follows from corollary 1 in [15]. So if $u(\cdot, v)$ has negative second derivative and positive third derivative with respect to the first argument for every v, then the optimal F is not third order stochastically dominated.

Proposition 1 is related to recent axiomatic work in non-linear utility theory, in particular Chew [2], and Fishburn [6, 7, 8]. There are two distinct approaches in this research, depending on whether transitivity of preferences is assumed [2, 7] or not [6, 8]. It is common in both cases to use a type of symmetry axiom which imposes restrictions on how indifference sets relate to one another, while the betweenness axiom imposes convexity on each indifference set (see the indifference sets in the probability simplices in Fig. 1). Of course, the additional restriction provides stronger results, essentially guaranteeing the skew-symmetry of a bilinear function ϕ: $D \times D \to \mathbb{R}$, which represents preferences by $\phi(p, q) > 0$ if and only if $p \succ q$ [6]. With transitivity $\phi(\cdot, \cdot)$ can be decomposed [7] and a weighted expected utility decomposition has been analyzed [2].

The results closest to my work are those of Chew [3] and Fishburn [7]. In [3] Chew has independently provided an implicit weighted utility characterization of preferences satisfying weak order, continuity and sub-

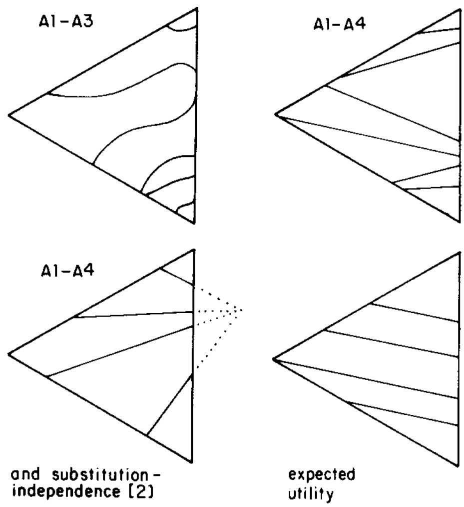

FIGURE 1

stitution axioms which, taken together, are equivalent to A1(a), A2 and A4. Preferences are represented by the solution of an implicit equation which has the form of a weighted utility function (cf. [2]) rather than the implicit expected utility structure in Proposition 1. Fishburn also does not require A1(b) and compactness of W, assuming instead countable boundedness (there exists a countable subset $\bar{D}$ of D such that for every $P \in D$ there is Q, $Q' \in \bar{D}$ with $Q \gtrsim P \gtrsim Q'$). Other than this his axioms are equivalent to A1, A2, and A4 (Continuity in [7] is A2 and Dominance is A4) giving [7, Theorem 1]: Countable boundedness, A1(a), A2, and A4 hold iff there exists a function f: $D \to R$ s.t. $P, Q \in D$, $P \succ Q$ iff $f(P) > f(Q)$ and $f(\alpha P + (1-\alpha)Q)$ is continuous and increasing (constant) in α if $P \succ Q (P \sim Q)$.

The representation in this paper is a more refined functional form, closer in structure to expected utility, admits a simple analysis of risk aversion and dominance, and has a simple uniqueness characterization.

Proposition 1 bears a formal resemblence to Fishburn's implicit characterization of a certainty equivalent functional m: $D \to \mathbb{R}$ [8]. $m(\cdot)$ is defined from $\int \phi(x, m(P))\, dP(x) = 0$, where ϕ is a skew symmetric, monotone function and W is an interval of the real line. However, the cancellation axiom in [8] is of the symmetry class, thus ϕ is skew symmetric while $u(\cdot, \cdot)$ may not be. Note that when W is restricted to a compact interval of $\mathbb{R}$, I can use Proposition 1 to provide a mean value representation. Given $u(w, v)$, normalized so that $u(\bar{w}, \cdot) = 1$, $u(\mathbf{w}, \cdot) = 0$, define $p(w)$ as the unique p which satisfies $w \sim (p, \bar{w};\ (1-p), \mathbf{w})$ and define $c(w, w') =$

$u(w, p(w')) - p(w')$. The certainty equivalent $M[F] \equiv \{w \in D \mid w \sim F\}$ satisfies $u(M[F], V[F]) = V[F]$ by (*) so we have $\int c(w, M[F])\, dF(w) = \int \{u(w, V[F]) - V[F]\}\, dF(w) = 0$. This shows how a generalized mean value without symmetry axioms can be derived using the approach of this paper. (Note that $c(\cdot, \cdot)$ may or may not be skew symmetric depending on whether or not the cancellation axiom is satisfied.)

Before going through the constructive proof, it is worthwhile to consider the intuition of the representation. A4 implies that indifference sets are convex. Since thick indifference sets are ruled out (by A4), we are left with indifference sets as hyperplanes. Recall that preferences of the expected utility type have parallel hyperplanes for indifference sets. Imagine now that given the indifference hyperplane, say $H(v)$, through the lottery $(v, \bar{w}; (1-v), \mathbf{w})$ we ignore all the other indifference sets and construct instead a collection of parallel hyperplanes. These can be taken to represent preferences satisfying the expected utility hypothesis and therefore there exists a function u_v (the subscript indicating the original hyperplane $H(v)$) which satisfies $\int u_v(\cdot)\, dF(\cdot) =$ the expected utility evaluation of F. If we set, as we are free to do with expected utility preferences, $u_v(\bar{w}) = 1$ and $u_v(\mathbf{w}) = 0$ then for $F = (v, \bar{w}; (1-v)\mathbf{w})$ we have $u_v(\bar{w})v + u_v(\mathbf{w})(1-v) = v$. Thus for any $F' \in H(v)$, which is an indifference set both for the original preferences and these artificial expected utility preferences, we know that $\int u_v(\cdot)\, dF'(\cdot) = v$. Doing this for indifference hyperplanes through points $(v, \bar{w}; (1-v), \mathbf{w})$ for every $v \in (0, 1)$ we get a collection of functions $u_v(w)$ which is exactly $u(w, v)$. The intuition of examining the expected utility extension of a given indifference hyperplane lies behind most of the subsequent results. A number of the proofs are done using the characterization (**). This is not restrictive and is only a choice of normalization.

Proof of Proposition 1. I will choose a normalization and prove the existence of a representation such as (**), and the uniqueness result will extend this to representations of the form (*). For any (p, w) with $w \neq \mathbf{w}$, $w \neq \bar{w}$, and $p \in (0, 1)$ the lottery $(p, \bar{w}; (1-p), \mathbf{w})$ is either: (i) strictly preferred to w; (ii) strictly worse than w, or (iii) indifferent to w. By solvability find (i) a $\beta \in (0, 1)$ s.t. $(\beta, \bar{w}; (1-\beta), w) \sim (p, \bar{w}; (1-p), \mathbf{w})$; or (ii) a $\gamma \in (0, 1)$ s.t. $(\gamma, \mathbf{w}; (1-\gamma), w) \sim (p, \bar{w}; (1-p), \mathbf{w})$. In case (i) set $u(w, p) = (p-\beta)/(1-\beta)$, in case (ii) $u(w, p) = p/(1-\gamma)$, and in case (iii) $u(w, p) = p$. For $w = \bar{w}$ set $u(\bar{w}, p) = 1$, $\forall p$; and for $w = \mathbf{w}$ set $u(\mathbf{w}, p) = 0$. Since $u(w, v)$ will be shown to be continuous on the open interval $(0, 1)$, extend the definition of $u(w, v)$ to the closed interval by continuity. Diagramatically (see Fig. 2) what has been done is: (a) construct the intersection of the indifference set through $(p, \bar{w}; (1-p), \mathbf{w})$ with the 2-dimensional simplex with vertices $(\mathbf{w}, w, \bar{w})$; (b) find the line parallel to this intersection going through the w vertex; this is the dashed line in the diagram;

(c) define the point at which that parallel line meets the $(\mathbf{w}, \bar{w})$ edge of the simplex as $u(w, p)$. This is exactly the value that expected utility preferences parallel to the hyperplane through $(p, \bar{w}; (1-p), \mathbf{w})$ would have assigned to the sure outcome w (if the values of $\bar{w}$ and $\mathbf{w}$ were normalized to 1 and 0).

The proof that (**) actually represents the preferences when using the constructed $u(\cdot, \cdot)$ will proceed in five steps:

(1) assigning a value to lotteries $(p, \bar{w}; (1-p), \mathbf{w})$,

(2) considering lotteries on the edges of $(\mathbf{w}, w, \bar{w})$ simplices,

(3) considering other two-outcome lotteries,

(4) lotteries in a $(\mathbf{w}, w, \bar{w})$ simplex,

(5) general simple lotteries.

(1) Let $V[p, \bar{w}; (1-p), \mathbf{w}] = p$, which obviously retains the preference ordering of such lotteries. Substituting in (**) gives $pu(\bar{w}, p) + (1-p)\, u(\mathbf{w}, p) = p$ as required.

(2) Consider $(\beta, \mathbf{w}; (1-\beta), w) \sim (p, \bar{w}; (1-p), \mathbf{w})$. By the previous step it is sufficient to show that $(1-\beta)\, u(w, p) + \beta u(\mathbf{w}, p) = p$. By construction $u(w, p) = p/(1-\beta)$ so $(1-\beta)\, p/(1-\beta) + \beta \cdot 0 = p$ as desired. A similar proof holds for lotteries on the $(w, \bar{w})$ edge.

(3) This stage in the proof shows that for lotteries of the type $(\alpha, w'; (1-\alpha), w'')$ with $w', w'' \in W$, if they are indifferent to, say $(p, \bar{w}; (1-p), \mathbf{w})$, then $\alpha u(w', p) + (1-\alpha)\, u(w'', p) = p$. Since the proof is a simple but lengthy geometric analysis it is provided in Appendix B.

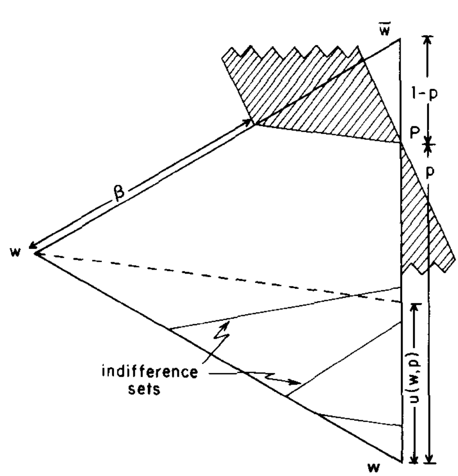

FIGURE 2

(4) Given $Q=(\mathbf{q}, \mathbf{w};\ q, w;\ \bar{q}, \bar{w}) \sim (p, \bar{w};\ (1-p), \mathbf{w})$ with $\mathbf{q}+q+\bar{q}=1$, examine Fig. 3 to see how Q is viewed as a mixture of the two lotteries on the edge of the simplex, which are indifferent to Q:

$$Q=\alpha(p, \bar{w};\ (1-p), \mathbf{w})+(1-\alpha)(t, \bar{w};\ (1-t)\ w)$$

and

$$Q \sim (p, \bar{w};\ (1-p), \mathbf{w}) \sim (t, \bar{w};\ (1-t), w).$$

I need to show that $u(\mathbf{w}, p)\mathbf{q}+u(w, p)q+u(\bar{w}, p)\bar{q}=p$. By the decomposition of Q, $\mathbf{q}=\alpha(1-p)$, $q=(1-\alpha)(1-t)$, and $\bar{q}=\alpha p+(1-\alpha)t$. Thus, $u(\mathbf{w}, p)q+u(w, p)q+u(\bar{w}, p)\bar{q} = \alpha[(1-p)\,u(\mathbf{w}, p)+pu(\bar{w}, p)]+(1-\alpha)[(1-t)\,u(w, p)+tu(\bar{w}, p)]=\alpha p+(1-\alpha)\,p=p$, since the first square brackets equal p by step (1) and the latter square brackets equal p by step (2).

(5) Given a simple measure P which assigns positive weights $p_1,\ldots, p_n$ to $w_1,\ldots, w_n$ and is indifferent to $(p_0, \bar{w};\ (1-p_0), \mathbf{w})$ it is necessary to prove that $\sum_{i=1}^{n} p_i u(w_i, p_0)=p_0$. Consider the simplex $\Delta \subset D$ which includes all measures over $w_1,\ldots, w_n$. The intersection of the indifference hyperplane through P with Δ (this intersection is denoted by H) is a compact convex subset of Δ, thus any point $h \in H$ can be written as a finite convex combination of extreme points of H. Therefore, $P=\sum_{i=1}^{m} \lambda_j Q_j$, where $Q_j \in H$ is a lottery assigning probability q_j to w' and $(1-q_j)$ to w'' (these are extreme points of H, where $w', w'' \in \Delta$). By step (3) above $\int u(w, p_0)\, dQ_j=p_0$. Therefore, $\int u(w, p_0)\, dP = \int u(w, p_0)\, d(\sum \lambda_j Q_j) = \sum \lambda_j \int u(w, p_0)\, dQ_j = \sum \lambda_j p_0=p_0$. ∎

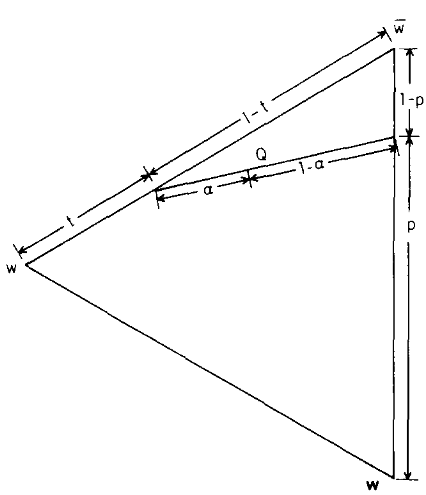

FIGURE 3

I now present the continuous representation theorem (Proposition 2), since the proofs of the properties and uniqueness results are identical for both representations and are provided in Section 3. Proposition 1 showed that there is a characterization similar to expected utility even when the independence axiom is weakened, for all simple measures on a compact consequence space. In order to get an integral representation theorem for more general measures we need more assumptions (just as in expected utility theory—see Fishburn [5, Chap. 3]). Rather than attempt to provide equivalent theorems for all possible extension results, only one approach of special interest is considered. It allows for a continuous "local utility" function $u(w, v)$ by assuming that preferences are continuous. This is in the spirit of Grandmont [9], adapted to the more general approach of this paper.

A2′. Continuity: The sets $\{P \in D\colon P \succsim P^*\}$ and $\{P \in D\colon P^* \succsim P\}$ for all $P^* \in D$ are closed (in the topology of weak convergence).

PROPOSITION 2. *Preferences over D satisfy* A1(a), A2′, A3, A4 *if and only if there exists $u(\cdot, \cdot)\colon W \times [0, 1] \to \mathbb{R}$ increasing in the preference ordering of W, continuous in both its arguments, such that $P \succ Q$ (resp. $P \sim Q$) $\Leftrightarrow V[P] > V[Q]$ (resp. $V[P] = V[Q]$), where $V[F]$ is defined implicitly as the unique $v \in [0, 1]$ that solves*

$$\int u(w, v)\, dF(w) = vu(\bar{w}, v) + (1 - v)\, u(\mathbf{w}, v). \qquad (*)$$

Furthermore, $u(w, v)$ is unique up to positive affine transformations which are continuous functions of v.

Proof. A1(b) is implied by compactness of D and A2′. Parts (1)–(4) of the proof are as before and only part (5) changes as below, where $\int u(\cdot, p)\, dQ(\cdot)$ is a continuous linear function of $Q \in D$ since the constructed $u(w, p)$ is continuous by A2′.

(5′) Given a lottery $F(\cdot) \sim (p, \bar{w};\ (1-p)\mathbf{w})$ show that $\int u(w, p)\, dF(w) = p$.

By Choquet's theorem ([14], pp. 19, 20) there exists a probability measure, say ν, on the indifference hyperplane H which includes $F(\cdot)$, s.t. ν represents F and is supported by the extreme points of H. An extreme point, say x, of H is one which can be represented only by the measure which assigns 1 to all Borel sets of H which include x, zero elsewhere. In this case the extreme points are those on simplex edges, i.e., of the type $(p, w';\ (1-p), w'')$. Thus the continuous linear function $U(Q) \equiv \int u(\cdot, p)\, dQ(\cdot)$ for $Q \in D$ satisfies $U(F) = \int_H U(\cdot)\, d\nu$, where $\nu(H \backslash S) = 0$ and

S is the set of all distributions on the indifference hyperplane which are also on simplex edges. Since for each s in S, $U(s)=p$ (by (2) and (3) above); this shows that $U(F)=p$. ∎

3. PROPERTIES OF THE CHARACTERIZATION

A. $u(w, v)$ *Is Increasing in* w

The proof that $u(w, v)$ is increasing in w relies on monotonicity. For any $v \in [0, 1]$ and $w \succ w'$ consider $P \equiv (v, \bar{w}; (1-v), \mathbf{w})$. *If* $w \succ P \succ w'$ *then* $u(w, v) > v > u(w', v)$ *by the construction of* $u(\cdot, \cdot)$. If $w \succ w' \succ P$ then by A2 find β and β' such that $(\beta, w; (1-\beta), \mathbf{w}) \sim P$ and $(\beta', w'; (1-\beta'), \mathbf{w}) \sim P$. Now, $\beta' > \beta$, since otherwise $P \sim (\beta, w; (1-\beta), \mathbf{w}) \succ (\beta, w'; (1-\beta), \mathbf{w}) \succsim (\beta', w'; (1-\beta'), \mathbf{w}) \sim P$ (where the strict preference follows from A3 and the weak preference can be derived using A4). Thus $u(w, v) = v/\beta$ is greater than $u(w', v) = v/\beta'$. The proof for the case when $P \succ w \succ w'$ is similar.

B. *Uniqueness of* $u(w, v)$ *Up to Continuous Positive Affine Transformations*

The proof of uniqueness up to affine trasformations includes two steps. First I show that for any function $g(w, v)$ increasing in w, no preference functional other than those assigning value p to lotteries $F_p \equiv (p, \bar{w}; (1-p), \mathbf{w})$ are represented by (*). Consider these distributions F_p and a possible preferrence functional $H[F_p]$. By substituting into (*), $g(\bar{w}, H[F_p])\, p + g(\mathbf{w}, H[F_p])(1-p) = g(\bar{w}, H[F_p])\, H[F_p] + g(\mathbf{w}, H[F_p])(1 - H[F_p])$. This implies that $p = H[F_p]$, since $g(\bar{w}, H[F_p]) \neq g(\mathbf{w}, H[F_p])$ by assumption.

The next stage asks whether, for a fixed preference function $V[\cdot]$, there exist transformations of $u(\cdot, \cdot)$ for which $V[F]$ is the solution to (*). Obviously $V[F]$ still solves (*) if we take a positive continuous affine transformation of $u(\cdot, \cdot)$. These are now shown to be the only acceptable transformations. Assume $\int h(w, p)\, dF(w) = h(\mathbf{w}, p)(1-p) + h(\bar{w}, p)\, p$ and $\int u(w, p)\, dF(w) = p$ so that $h(\cdot, \cdot)$ correctly solves $V[F] = p$. Define $b(p) = h(\mathbf{w}, p)$, $a(p) = [h(\bar{w}, p) - h(\mathbf{w}, p)]$, and $g(w, p) = a(p)\, u(w, p) + b(p)$. I now show that $g(w, p) = h(w, p)$, so any solution $h(w, p)$ which solves (**) is a generalized affine transformation of $u(w, p)$. For any $F \sim (p, \bar{w}; (1-p), \mathbf{w})$, $\int g(w, p)\, dF(w) = \int [a(p)\, u(w, p) + b(p)]\, dF(w) = [h(\bar{w}, p) - h(\mathbf{w}, p)] \int u(w, p)\, dF(w) + h(\mathbf{w}, p) = [h(\bar{w}, p) - h(\mathbf{w}, p]\, p + h(\mathbf{w}, p) = \int h(w, p)\, dF(w)$. Now consider $F_w = (\beta, w; (1-\beta), \mathbf{w})$ or $F_w = (\gamma, w; (1-\gamma)\bar{w})$ such that $F_w \sim F$, one of which exists by solvability. Then either $h(w, p)\beta + h(\mathbf{w}, p)(1-\beta) = g(w, p)\beta + g(\mathbf{w}, p)(1-\beta)$ or $h(w, p)\gamma + h(\bar{w}, p)(1-\gamma) = g(w, p)\gamma + g(\bar{w}, p)(1-\gamma)$, but since by construction $h(\bar{w}, p) = g(\bar{w}, p)$ and $h(\mathbf{w}, p) = g(\mathbf{w}, p)$ this implies $h(w, p) = g(w, p)$.

C. *Uniqueness of the Implicit Solution*

Since $V[F]$ is defined implicitly, it is necessary to show that the solution to the implicit function is unique. This is done by considering the expected utility extension of these preferences. Assume the solution to (**) is not unique, i.e., in addition to the correct solution v, there exists $\hat{v} \in [0, 1]$, $\hat{v} \neq v$ such that $\int u(w, v)\, dF(w) = v$, $\int u(w, \hat{v})\, dF(w) = \hat{v}$, and $\int u(w, \hat{v})\, d\hat{F}(w) = \hat{v}$, where $\hat{v}$ is the correct solution for $\hat{F}$. (Solutions where $\hat{v} \notin [0, 1]$ can be ignored since, even if they solve equation (**) they lie outside the range of permissible values—recall that $v \in [0, 1]$.) Holding $\hat{v}$ constant consider $\hat{u}(w) = u(w, \hat{v})$ as a Bernoulli utility function which defines expected utility preferences through $\hat{F}$ and $(\hat{v}, \bar{w};\ (1 - \hat{v}), \mathbf{w})$ but not through F (the latter by assumption that $\hat{v}$ is not the correct solution for F). However, by assumption also $\int \hat{u}(\cdot)\, d\hat{F}(\cdot) = \hat{v} = \int \hat{u}(\cdot)\, dF(\cdot)$, implying that F and $\hat{F}$ do lie in the same indifference hyperplane.

D. *For Every $w \in W$, $u(w, v)$ Is Continuous in v on the Open Interval* (0, 1)

First fix w not indifferent to $\bar{w}$, and consider the simplex with vertices $\mathbf{w}$, w, and $\bar{w}$. Let $B(w) = \{v \mid (v, \bar{w};\ (1 - v), \mathbf{w}) \succsim w\}$ and note that $B(w)$ is a closed interval from some $\bar{v}$ to 1. The function $\beta(v)$ (which was used in the construction of $u(w, v)$) is defined as the solution of $(\beta, \bar{w};\ (1 - \beta), w) \sim (v, \bar{w};\ (1 - v), \mathbf{w})$ for any $v \in B(w)$. Clearly $\beta(\bar{v}) = 0$, $\beta(1) = 1$, and $\beta(\cdot)$ is an increasing function (otherwise two indifference lines in the simplex will cross). I now show that $\beta(\cdot)$ is continuous. If not then there exists $v_n \uparrow v$ with $\beta(v) > \lim \beta(v_n)$. So for $\hat{\beta}$ satisfying $\beta(v) > \hat{\beta} > \lim \beta(v_n)$ it is clear that $(v, \bar{w};\ (1 - v), \mathbf{w}) \succ (\hat{\beta}, \bar{w};\ (1 - \hat{\beta}), w) \succ (v_n, \bar{w};\ (1 - v_n), \mathbf{w})$ for every n. Hence there exists a (unique) $\hat{v}$ such that $(\hat{\beta}, \bar{w};\ (1 - \hat{\beta}), w) \sim (\hat{v}, \bar{w};\ (1 - \hat{v}), \mathbf{w})$ and $\hat{v} \in (v_n, v)$ for every n. But this cannot be satisfied since $v_n \uparrow v$. So the assumption that $\beta(\cdot)$ is not continuous leads to a contradiction. Recall that by construction $u(w, v)$ is equal to $u(w, v) = (v - \beta(v))/(1 - \beta(v))$ for $v \in (\bar{v}, 1)$ and equal to $\bar{v}$ when $v = \bar{v}$, so $u(w, v)$ is continuous for $v \in [\bar{v}, 1)$. A similar proof shows that $u(w, v)$ is continuous for $v \in (0, \bar{v}]$. For $w \sim \bar{w}$ (resp. $w \sim \mathbf{w}$) monotonicity implies that $u(w, v) = u(\bar{w}, v) = 1$ (resp. $u(w, v) = u(\mathbf{w}, v) = 0$).

Continuity is necessary to avoid indifference sets which do not separate the simplex into two disconnected sets. For example, in the simplex with vertices $\mathbf{w}$, w and $\bar{w}$, if $u(w, v) = \frac{1}{4}$ for $v < \frac{1}{2}$ and $u(w, v) = \frac{1}{2}$ for $v \geqslant \frac{1}{2}$, there would be indifference lines which end inside of the simplex.

4. Extending Local Properties

This section relates this representation of preferences to Machina's [12] work on non-expected utility preferences. If all preferences satisfying

A1–A4 had a preference functional $U[F]$ which was everywhere Frechet differentiable, then Machina's extension results (that local monotonicity and risk aversion everywhere in D imply global monotonicity and risk aversion) would go through. It is shown below that this is not true, in particular it is shown how to construct a set of indifference lines in the simplex which have no differentiable preference functional representation. However, an alternative approach to extending local results is presented. Rather than examine the first-order approximation to the preference functional (which may not be smooth), if the indifference sets are smooth manifolds then the first order approximation to an indifference curve can be taken and extended to parallel hyperplanes, giving an expected utility approximation. For preferences considered in this paper such an extension is simple. It is shown that for $W \subset \mathbb{R}$ if $u(w, \cdot)$ is increasing in w, then P first order stochastically dominates Q, if and only if $V[P] > V[Q]$. (Note that the property shown in the previous section is that $u(w, v)$ is increasing in w with respect to the preference order on W. Any conclusion on stochastic dominance requires that this induced order is the natural order on the reals, i.e., $w \succ w'$ if and only if $w > w'$.) Furthermore it is proven that if $u(w, v)$ is concave in w for every v then the individual is averse to mean preserving increases in risk.

Let f: $[0, \frac{1}{2}] \to [0, a]$ for some $a \in (\frac{1}{2}, 1)$ be continuous, strictly increasing with derivative zero a.e. (see Billingsley [1, Ex. 31.1]). Define the indifference sets in a simplex as in Fig. 4. Let $V(\cdot)$ be a functional representing these preferences, so that $V(1, f(y)) = V(0, y)$ for $y \in [0, \frac{1}{2}]$, where the second argument indicates the distance along the simple edge from the lower vertex, and the first argument indicates which edge (1 for the lower sloped edge, and 0 for the vertical edge). If $V(\cdot)$ is differentiable then $V_2(0, y) = V_2(1, f(y))\, f'(y)$ wherever $f(\cdot)$ is differentiable. Hence $V_2(0, y) = 0$ a.e., implying that $V(0, y)$ is constant for $y \in [0, \frac{1}{2})$ (note that

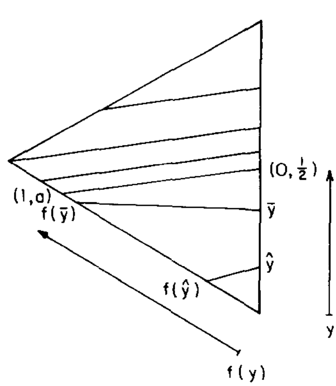

FIGURE 4

$V(0, \cdot)$ is absolutely continuous since it is everywhere differentiable and monotone). However, since these preferences are by assumption strictly increasing along the vertical edge of the simplex, we have a contradiction. Therefore V cannot be differentiable.

PROPERTY 1. *The following statements are equivalent*:

(a) *For any P, $Q \in D$ if P stochastically dominates Q then P is preferred to Q.*

(b) *$u(w, v)$ is increasing in w.*

Proof. Assume that P first order stochastically dominates Q, while p and q which solve $\int u(w, p)\, dP(w) = p$ and $\int u(w, q)\, dQ(w) = q$ satisfy $p < q$. The indifference hyperplane through Q separates D into two convex sets:

$$\mu \in U \Rightarrow \int u(w, q)\, d\mu(w) \geqslant \int u(w, q)\, dQ(w) = q$$

$$v \in L \Rightarrow \int u(w, q)\, dv(w) \leqslant \int u(w, q)\, dQ(w) = q.$$

Since P stochastically dominates Q, $P \in U$. If $p < q$ then $(p, \bar{w};$ $(1-p), \mathbf{w}) \in L$, since $pu(\bar{w}, q) + (1-p)\, u(\mathbf{w}, q) = p < q$. So the convex indifference surface through P and $(p, \bar{w}; (1-p), \mathbf{w})$ lies both above and below the separating hyperplane, thus two indifference sets intersect, obviously leading to a contradiction. The converse is straightforward. ▮

PROPERTY 2. *Concavity of $u(w, v)$ in w implies risk aversion (in the sense that the individual is weakly averse to mean preserving increases in risk).*

Proof. Assume that $u(w, v)$ is concave in w for every v, and that G differs from F by a mean preserving increase in risk. Hence $\int u(w, v)\, d[G(w) - F(w)] < 0$. Let p and q solve $\int u(w, p)\, dF(w) = p$ and $\int u(w, q)\, dG(w) = q$. Then $q = \int u(w,q)\, d[F(w) + (G(w) - F(w))] <$ $\int u(w, q)\, dF(w)$, so F lies above the indifference hyperplane through q. If $q > p$ then $(p, \bar{w}; (1-p), \mathbf{w})$ lies below the indifference hyperplane through q. But by betweenness the indifference set which includes $F \sim (p, \bar{w};$ $(1-p), \mathbf{w})$ is convex, intersecting the separating indifference hyperplane through q, leading to a contradiction; hence $q < p$. ▮

Appendix

A. *The Representation Theorem without Monotonicity*

Proposition A.1. *Preferences over* D_0 *(resp. D) satisfy* A1, A2, *and* A4 *(resp.* A1(a), A2′, *and* A4) *if and only if there exists* $u(\cdot, \cdot)$: $W \times [0, 1] \to \mathbb{R}$, *continuous in the second argument (resp. continuous in both arguments), such that* $P > Q \Leftrightarrow V[P] > V[Q]$ *and* $P \sim Q \Leftrightarrow V[P] = V[Q]$, *where* $V[F]$ *is defined implicitly as the unique* $v \in [0, 1]$ *that solves*

$$\int u(w, v)\, dF(w) = vu(\bar{w}, v) + (1 - v)\, u(\mathbf{w}, v). \qquad (*)$$

Furthermore, $u(w, v)$ *is unique up to positive affine transformations which are continuous in* v.

The proof follows essentially the same lines as the proofs of Propositions 1 and 2. However, monotonicity of $u(w, v)$ in w cannot be proven without A3 (see Sect. 3A).

B. *Step* 3 *in Proving the Representation Theorem*

For any distribution $R = (\theta, w'; (1-\theta), w'')$ consider the 3-dimensional simplex with vertices $(\mathbf{w}, \bar{w}, w', w'')$, where without loss of generality assume $w'' > w'$. By A2 find p such that $R \sim (p, \bar{w}; (1-p), \mathbf{w})$, where this last lottery is the point B in the 3-dimensional simplex.

Construct the indifference hyperplane through R (see Fig. 5). I want to show that $\theta u(w', p) + (1-\theta)\, u(w'', p) = p$. Define C as the lottery $(\gamma, \mathbf{w};$

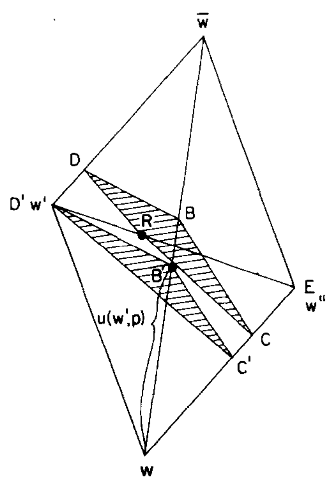

Figure 5

$(1-\gamma), w'') \sim B$, and E as the degenerate lottery w''. Thus, $CE = \gamma$ and $u(w'', p) = p/(1-\gamma)$ by case (ii) on p. 9, with w replaced by w''. Using trigonometric identities based on equilateral triangles with edges of length normalized to 1, using also the lengths γ, θ, and $1-\theta$ it will be shown that $u(w', p) = ((\theta-\gamma)/\theta) \cdot p/(1-\gamma)$ giving the desired result. Take a parallel shift of the indifference plane through R, D, and C such that the new plane intersects w'. This new plane intersects the $(\mathbf{w}, w'')$ edge at point C' and the $(\mathbf{w}, \bar{w})$ edge at B'. Recall that by definition $u(w', p)$ is the length of the segment between B' and $\mathbf{w}$. Therefore $u(w', p)$ can be found from the length of RC in triangle REC, where $RE = \theta$, $CE = \gamma$ and $\measuredangle \mathrm{REC} = 60°$, giving $RC = (\theta^2 + \gamma^2 - \gamma\theta)^{1/2}$. Then $\cos \measuredangle CRE = (2\gamma - \theta)/2(\theta^2 + \gamma^2 - \gamma\theta)^{1/2}$, and $\cos \measuredangle CR = 2(\theta - \gamma)/2(\theta^2 + \gamma^2 - \gamma\theta)^{1/2}$. By examining the trapezoid with corners (w', R, C, C') one can see that length $CC' = (1-\theta)\,\gamma/\theta$ and thus length $\mathbf{w}C' = (\theta - \gamma)/\theta$. Looking now at the $(\bar{w}, \mathbf{w}, w'')$ simplex, $\sin \measuredangle \mathbf{w}BC = \sin 60(1-\gamma)/BC$ and $\sin \measuredangle BC\mathbf{w} = p(\sin 60)/BC$. Thus, since $u(w', p) = B'\mathbf{w} = \mathbf{w}C'(\sin BC\mathbf{w})/(\sin \mathbf{w}BC)$, it has been shown that $u(w', p) = (\theta - \gamma)\, p/\theta(1-\gamma)$ as desired. ∎

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